COMPUTER GRAPHICS

THIRD CLASS

University of Diyala/ College of education for pure science / Computer science department

Dr. Adil I. KHALIL

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3. Two-Dimensional Geometric Transformation (Part 1-2)

Transformation is the ability to simulate the movement and the manipulation of objects in the plane, or is mapping appoint/vector to another point/vector. These processes are described in terms of :

1. Translation
2. Scaling
3. Rotation
4. Reflection
5. Shearing

Each of these transformations is used to generate a new point ($\bar{x}, \bar{y}$) from the coordinates of a point (x,y) in the original picture description. If the original definition includes a line, it suffices to apply the transformation to the endpoints of the line and display the line between the two transformed endpoints.

There are two types of transformations:

1. **Modeling Transformation** this transformation alters the coordinate values of the object. Basic operations are scaling, translation, rotation and, combination of one or more of these basic transformations. Examples of these transformations can be easily found in any commercial CAD software. For instance, AutoCAD uses SCALE, MOVE, and ROTATE commands for scaling, translation, and rotation transformations, respectively.
2. **Visual Transformation**: In this transformation there is no change in either the geometry or the coordinates of the object. A copy of the object is placed at the desired sight, without changing the coordinate values of the object. In AutoCAD, the ZOOM ,and the motion of a car in a scene, we can keep the car fixed while moving the background scenery examples of visual transformation.

Basic Modeling Transformations: There are three basic modeling transformations: Scaling, Translation, and Rotation. Other transformations, which are modification or combination of any of the basic transformations, are Shearing, Mirroring, Copy, etc.

Let us look at the procedure for carrying out basic transformations, which are based on matrix operation. A transformation can be expressed as

$$[P^*] = [P]\ [T]$$

where, $[P^*]$ is the new coordinates matrix

[P] is the original coordinates matrix, or points matrix

[T] is the transformation matrix

With the z-terms set to zero, the P matrix can be written as,

$$[P] = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 0 \end{pmatrix}$$

The size of this matrix depends on the geometry of the object, e.g., a point is defined by a single set of coordinates (x_1, y_1, z_1), a line is defined by two sets of coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), etc. Thus a point matrix will have the size 1x3, line will be 2x3, etc. A transformation matrix is always written as a 4x4 matrix, with a basic shape shown below,

$$[T] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Values of the elements in the matrix will change according to the type of transformation being used, as we will see shortly. The transformation matrix changes the size, position, and orientation of an object, by mathematically adding, or multiplying its coordinate values. We will now discuss the mathematical procedure for translation, scaling, and rotation transformations.

Homogeneous Coordinates

Before proceeding further, we should review the concept of homogeneous coordinate system. Since the points matrix has three columns for the x, y, and z values, and a transformation matrix is always 4x4 matrix, the two matrices are incompatible for multiplication. A matrix multiplication is compatible only if the number of columns in the first matrix equals the number of row in the second matrix. For this reason, a points matrix is written as,

$$[P] = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & z_n & 1 \end{pmatrix}$$

Here, we have converted the Cartesian coordinates into homogeneous coordinates by adding a 4^{th} column, with unit value in all rows. When a fourth column, with values of 1 in each row, is added in the points matrix, the matrix multiplication between the [P] and [T] becomes compatible. The values (x_1, y_1, z_1, 1) represent the coordinates of the point (x_1, y_1, z_1), and the coordinates are called as homogeneous coordinates. In our subsequent discussion on transformation, we will use homogeneous coordinates.

1- Translation

In translation, every point on an object translates exactly the same distance. The effect of a translation transformation is that the original coordinate values increase or decrease by the amount of the translation along the x, y, and z-axes. For example, if line A(2,4), B(5,6) is translated 2 units along the positive x axis and 3 units along the positive y axis, then the new coordinates of the line would be A'(2+2, 4+3), B'(5+2, 6+3) or A'(4,7), B'(7,9).

The *transformation matrix has the form:*

$$[T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

where, x and y are the values of translation in the x and y direction, respectively. For translation transformation, the matrix equation is

$[P^*] = [P]\ [T_t]$

where, $[T_t]$ is the translation transformation matrix.

Example *: Translate the rectangle (2,2), (2,8), (10,8), (10,2) 2 units along x-axis and 3 units along y-axis.*

Solution: Using the matrix equation for translation, we have [P*] = [P] $[T_t]$, substituting the numbers, we get

$$[P^*] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 8 & 0 & 1 \\ 10 & 8 & 0 & 1 \\ 10 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 & 0 & 1 \\ 4 & 11 & 0 & 1 \\ 12 & 11 & 0 & 1 \\ 12 & 5 & 0 & 1 \end{pmatrix}$$

Note that the resultant coordinates are equal to the original x and y values plus the 2 and 3 units added to these values, respectively.

2- Scaling

Scaling is the process of expanding or compressing the dimensions of an object (changing the size of an object). There are two types of scaling transformations: **uniform** and **non-uniform**. In the uniform scaling, the coordinate values change uniformly along the x, y, and z coordinates, whereas, in non- uniform scaling, the change is not necessarily the same in all the coordinate directions, In scaling transformation, the original coordinates of an object are multiplied by the given scale factor (SF).

If SF (scale factor) > 1 then the object is enlarged
If SF (scale factor) < 1 then the object is compressed
If SF (scale factor) = 1 then the object is unchanged

1.1. Uniform Scaling

For uniform scaling, the scaling transformation matrix is given as

$$[T] = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here, ***s*** is the scale factor.

1.2. Non-Uniform Scaling

Matrix equation of a non-uniform scaling has the form:

$$[T] = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where, s_x, s_y, s_z are the scale factors for the x, y, and z coordinates of the object.

Example 1: If the triangle A(1,1), B(2,1), C(1,3) is scaled by a factor 2, find the new coordinates of the triangle.
Solution: Writing the points matrix in homogeneous coordinates, we have

$$[P] = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix}$$

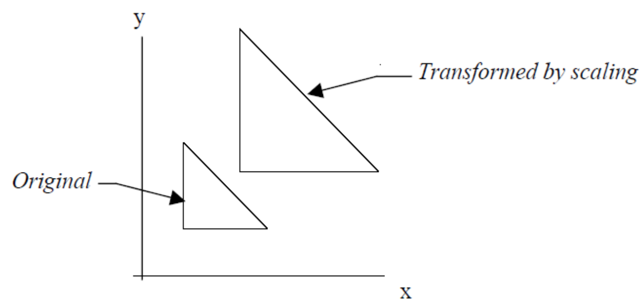
and the scaling transformation matrix is,

$$[T_s] = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The new points matrix can be evaluated by the equation

$[P^*] = [P]\ [T]$,

$$P^* = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 4 & 2 & 0 & 1 \\ 2 & 6 & 0 & 1 \end{pmatrix}$$



Note that the new coordinates represent the original value times the scale factor. The old and the new positions of the triangle are shown in the figure above.

Notice that after a scaling transformation is performed, the new object is located at a different position relative to the origin. In fact, in scaling transformation the only point that remains fixed is the origin.

If we want to let one point of an object that remains at the same location (fixed), scaling can be performed by three steps:

1- Translate the fixed point to the origin, and all the points of the object must be moved the same distance and direction that the fixed point moves.

2- Scale the translated object from step one

3- Back translate the scaled object to its original position

Example 2: Scale the rectangle (12,4),(20,4),(12,8),(20,8) with SX=2,SY=2, so the point (12,4) being the fixed point.

Solution:

1- Translate the object with TX= -12 and TY= - 4 so the point (12,4) lies on the origin

(12,4) → (0,0)
(20,4) → (8,0)
(12,8) → (0,4)
(20,8) → (8,4)

2- Scale the object by SX=2 and SY=2

(0,0) → (0 , 0)
(8,0) → (16, 0)
(0,4) → (0 , 8)
(8,4) →(16 , 8)

3- Back translate the scaled object with TX= 12 and TY= 4

(0, 0) → (12 , 4)
(16, 0) → (28 , 4)
(0, 8) → (12 ,12)
(16,8) → (28 ,12)

3- Rotation

We will first consider rotation about the z-axis, which passes through the origin (0,0,0), since it is the simplest transformation for understanding the rotation transformation. Rotation about an arbitrary axis, other than an axis passing through the origin, requires a combination of three or more transformations, as we will see later.

When an object is rotated about the z-axis, all the points on the object rotate in a circular arc, and the center of the arc lies at the origin. Similarly, rotation of an object about an arbitrary axis has the same relationship with the axis, i.e., all the points on the object rotate in a circular arc, and the center of rotation lies at the given point through which the axis is passing.

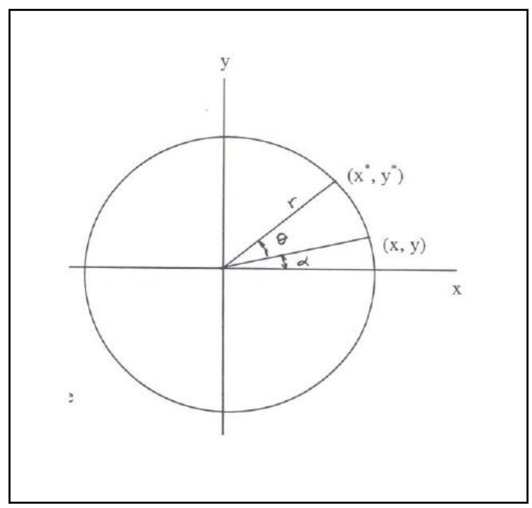
In rotation, the object is rotated Ø (theta) about the origin. The convention is that the direction of rotation is counterclockwise if Ø is a positive angle and clockwise if Ø is a negative angle.

Derivation of the Rotation Transformation Matrix (about the origin)

Using trigonometric relations, as given below, we can derive the rotation transformation matrix. Let the point P(x, y) be on the circle, located at an angle α, as shown. If the point P is rotated an additional angle θ, the new point will have the coordinates (x*, y*). The angle and the original coordinate relationship is found as follows.

$$\left.\begin{aligned} x &= r\cos\alpha \\ y &= r\sin\alpha \end{aligned}\right\}\quad \text{Original coordinates of point P.}$$

$$\left.\begin{aligned} x^* &= r\cos(\alpha+\theta) \\ y^* &= r\sin(\alpha+\theta) \end{aligned}\right\}\quad \text{The new coordinates.}$$



where, α is the angle

between the line joining the initial position of the point and the x-axis, and θ is the angle between the original and the new position of the point.

Using the trigonometric relations, we get,

$$x^* = r(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = x\cos\theta - y\sin\theta$$
$$y^* = r(\cos\alpha\sin\theta + \sin\alpha\cos\theta) = x\sin\theta + y\cos\theta$$

In matrix form we can write these equations as

$$[x * y *] = [xy]\begin{bmatrix} cos\emptyset & sin\emptyset \\ -sin\emptyset & cos\emptyset \end{bmatrix}$$

In general, the points matrix and the transformation matrix given in equation above are re-written as:

$$[x^* \quad y^* \quad 0 \quad 1] = [x \quad y \quad 0 \quad 1]\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, a point or any object can be rotated about the z-axis (in 2-D) and the new coordinates of the object found by the product of the points matrix and the rotation matrix derived here.

Rotation of an Object about an Arbitrary Axis (about a specific point)

Rotation of a geometric model about an arbitrary axis, other than any of the coordinate axes, involves several rotational and translation transformations. When we rotate an object about the origin (in 2-D), we in fact rotate it about the z-axis. Every point on the object rotates along a circular path, with the center of rotation at the origin. If we wish to rotate an object about an arbitrary axis, which is perpendicular to the xy-plane, we will have to first translate the axis to the origin and then rotate the model, and finally, translate so that the axis of rotation is restored to its initial position. If we erroneously use the equation above directly, to rotate the object about a fixed axis, and skip the translation of this point to the origin, we will in fact end up rotating the object about the z-axis, and not about the fixed axis.

Thus, the rotation of an object about an arbitrary axis, involves three steps:

Step 1: Translate the fixed axis so that it coincides with the z-axis

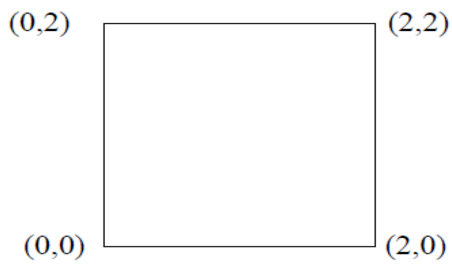
Step 2: Rotate the object about the axis

Step 3: Translate the fixed axis back to the original position.

Note: When the fixed axis is translated, the object is also translated. The axis and the object go through all the transformations simultaneously.

We will now illustrate the above procedure by the following example.

Example 3**: Rotate the rectangle (0,0), (2,0), (2, 2), (0, 2) shown below, 30^o** about its centroid and find the new coordinates of the rectangle.



Solution: Centric of the rectangle is at point (1, 1). We will first translate the centered to the origin, then rotate the rectangle, and finally, translate the rectangle so that the centered is restored to its original position.

1. Translate the centered to the origin: The matrix equation for this step is

$$[P^*]_1 = [P]\,[T_t],\ \text{where}\quad [P] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

$$\text{and}\qquad [T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$

2. **Rotate the Rectangle 30^o About the z-axis:** The matrix equation for this step is given as

$[P^*]_2$ = $[P^*]_1$ $[T_r]$, where, $[P^*]_1$ is the resultant points matrix obtained in step 1, and $[T_r]$ is the rotation transformation, where θ = 30^o. The transformation matrix is,

$$[T_r]_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .866 & .5 & 0 & 0 \\ -.5 & .866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. **Translate the Rectangle so that the Centerior Lies at its Original Position:** The matrix equation for this step is

$$[P^*]_3 = [P^*]_2 [_{T\text{-}t}],$$

where $[_{T\text{-}t}]$ is the reverse translation matrix, given as

$$[T_{-t}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Now we can write the entire matrix equation that combines all the three steps outlined above. The equation is,

$[P^*] = [P] [T_t] [T_r] [T_{-t}]$

Substituting the values given earlier, we get,

$$[P^*] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30^0 & \sin 30^0 & 0 & 0 \\ -\sin 30^0 & \cos 30^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \text{x}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.6340 & -0.3660 & 0 & 1 \\ -0.3660 & 1.3660 & 0 & 1 \\ 1.3660 & 2.3660 & 0 & 1 \\ -0.3660 & 1.3660 & 0 & 1 \end{pmatrix}$$

The first two columns represent the new coordinates of the rotated rectangle.

Note: Rotation in clockwise direction :In order to rotate in clockwise direction, the matrix will be

$$[x^* \quad y^* \quad 0 \quad 1] = [x \quad y \quad 0 \quad 1] \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combined Transformations

Most applications require the use of more than one basic transformation to achieve desired results. As stated earlier, scaling with an arbitrarily fixed point involves both scaling and translation. And rotation around a given point, other than the origin, involves rotation and translation. We will now consider these combined transformations.

Scaling with an arbitrary point

In uniform scaling, all points and their coordinates are scaled by a factor s. Therefore, unless the fixed point is located at (0, 0), it will be moved to a new location with coordinates s-times x and s-times y. To scale an object about a fixed point, the fixed point is first moved to the origin and then the object is scaled. Finally, the object is translated or moved so that the fixed point is restored to its original position. The transformation sequence is,

$$[P^*] = [P]\,[T_t]\,[T_s]\,[T_{-t}]$$

where, $[T_t]$ is the translation transformation matrix, for translation of the fixed point to the origin, $[T_s]$ is the scaling transformation matrix, and $[T_{-t}]$ is the reverse translation matrix, to restore the fixed point to its original position.

Note: The order of matrix multiplication progresses from left to right and the order should not be changed.

The three transformation matrices $[T_t]$ $[T_s]$ $[T_{-t}]$ can be concatenated to produce a single transformation matrix, which uniformly scales an object while keeping the pivot point fixed.

Thus, the resultant, concatenated transformation matrix for scaling is,

$$[T_s]_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ x-sx & y-sy & 0 & 1 \end{pmatrix}$$

The concatenated equation can be used directly instead of the step-by-step matrix solution. This form is preferable when writing a CAD program.

Example : Given the triangle, described by the homogeneous points matrix below, scale it by a factor 3/4, keeping the centroid in the same location. Use (a) separate matrix operation and (b) condensed matrix for transformation.

$$[P] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 5 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{pmatrix}$$

Solution

(a) The centroid of the triangle is at,

x = (2+2+5)/3 = 3, and y = (2+5+5)/3 = 4 or the centroid is C(3,4).

We will first translate the centroid to the origin, then scale the triangle, and finally translate it back to the centroid. Translation of triangle to the origin will give,

$$[P^*]_1 = [P]\,[T_t] = \begin{pmatrix}2&2&0&1\\2&5&0&1\\5&5&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-3&-4&0&1\end{pmatrix} = \begin{pmatrix}-1&-2&0&1\\-1&1&0&1\\2&1&0&1\end{pmatrix}$$

Scaling the triangle, we get,

$$[P^*]_2 = [P^*]_1\,[T_s] = \begin{pmatrix}-1&-2&0&1\\-1&1&0&1\\2&1&0&1\end{pmatrix}\begin{pmatrix}.75&0&0&0\\0&.75&0&0\\0&0&.75&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}-0.75&-1.5&0&1\\-0.75&0.75&0&1\\1.5&0.75&0&1\end{pmatrix}$$

Translating the triangle so that the centroid is positioned at (3, 4), we get

$$[P^*] = [P^*]_2\,[T_{-t}] = \begin{pmatrix}-.75&-1.5&0&1\\-.75&.75&0&1\\1.5&.75&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\3&4&0&1\end{pmatrix} = \begin{pmatrix}2.25&2.5&0&1\\2.25&4.75&0&1\\4.5&4.75&0&1\end{pmatrix}$$

(**b**) The foregoing set of three operations can be reduced to a single operation using the condensed matrix with x = 3, and y = 4. See page 12.

$$[P^*] = [P]\,[T_{cond}] = \begin{pmatrix}2&2&0&1\\2&5&0&1\\5&5&0&1\end{pmatrix}\begin{pmatrix}0.75&0&0&0\\0&0.75&0&0\\0&0&0.75&0\\3-0.75(3)&4-0.75(4)&0&1\end{pmatrix}$$

$$= \begin{pmatrix}2.25&2.5&0&1\\2.25&4.75&0&1\\4.5&4.75&0&1\end{pmatrix}$$