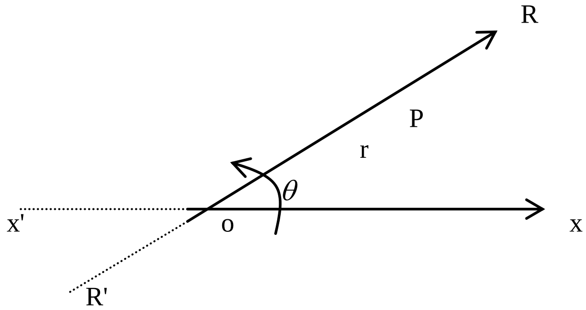
CHAPTER Seven
Polar Plan

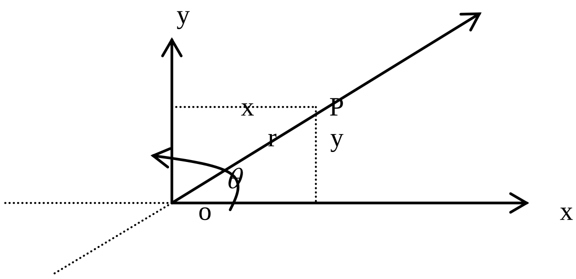
Polar coordinates

Let O be a fixed point, OX a fixed straight line.
Let P be any point on a line R'OR drawn through O in a direction making an angle θ with OX.



Let r=OP, and let r be positive or negative according as P is on the same side of O as R or R'.
Then (r,θ) are called the polar coordinates of the point P , O is called the pole, OX the initial line, OP the radius vector, and θ the vectorial angle of the point P.
When the point P describes a curve, its polar coordinates (r,θ) are connected by an equation, which is called the polar equation of the curve.

Relation between Polar and Cartesian Coordinates



Let (x, y) be the Cartesian coordinates of the point P referred to OX, OY as coordinate axes, and let (r,θ) be its polar coordinates referred to O as pole and OX as initial line. Then x and y are given in terms of r and θ by the equations

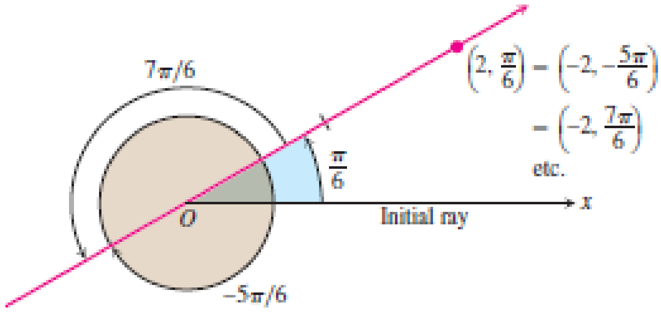
$$x = r\cos\theta, \qquad y = r\sin\theta; \qquad (1)$$

While r and p are given in terms of x and y by

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x} \qquad (2)$$

By means of (1) and (2) we can transform a Cartesian equation in to a polar equation, or vice versa.
Ex(1) **:-** Find all the polar coordinates of the point $p\left(2, \pi/6\right)$

Solution: We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $p(2,\pi/6)$. We then find the angles for the other coordinate pairs of *P* in which r=2 and r=-2.



For r=2, the complete list of angles is $\frac{\pi}{6},\frac{\pi}{6}\pm 2\pi,\frac{\pi}{6}\pm 4\pi,\frac{\pi}{6}\pm 6\pi,\ldots$

For r=-2, the angles are $-\frac{5\pi}{6},-\frac{5\pi}{6}\pm 2\pi,-\frac{5\pi}{6}\pm 4\pi,-\frac{5\pi}{6}\pm 6\pi,\ldots$

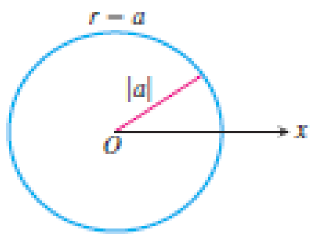
The corresponding coordinate pairs of P are $\left(2,\frac{\pi}{6}+2n\pi\right),\quad n=0,\pm1,\pm2,\ldots$

and $\left(-2,-\frac{5\pi}{6}+2n\pi\right),\quad n=0,\pm1,\pm2,\ldots$

when n=0, the formulas give $(2,\pi/6)$ and $(-2,-5\pi/6)$. When n=1, they give $(2,13\pi/6)$ and $(-2,7\pi/6)$, and so on.

Polar Equations and Graph

If we hold *r* fixed at a constant value $r=a\neq 0$, the point $P(r,\theta)$ will lie $|a|$ units from the origin *O*. As θ varies over any interval of length 2π, *P* then traces a circle of radius $|a|$ centered at *O* Figure.



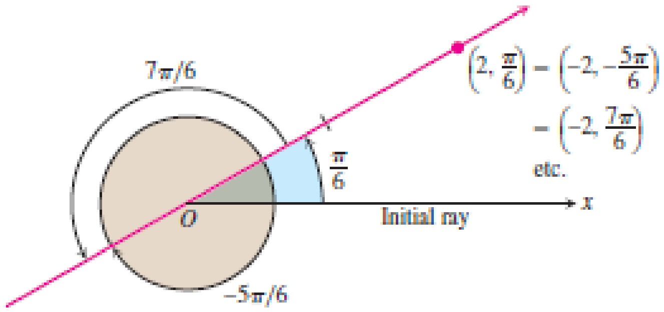
If we hold θ fixed at a constant value $\theta=\theta_0$ and let *r* vary between $-\infty$ and ∞, the point $P(r,\theta)$ traces the line through *O* that makes an angle of measure θ_0 with the initial ray.

Equation

$r=a$ Circle radius $|a|$ centered at O.

$\theta=\theta_0$ Line through O making an angle θ_0 with the initial ray.

Ex(2) :- (a) $r=1$ and $r=-1$ are equations for the circle of radius 1 centered at O.
(b) $\theta=\pi/6, \theta=7\pi/6$ and $\theta=-5\pi/6$ are equations for the line in Figure



Equations of the form $r=a$ and $\theta=\theta_0$ can be combined to define regions, segments, and rays.

Ex(3) **:-** Graph the sets of points whose polar coordinates satisfy the following conditions.

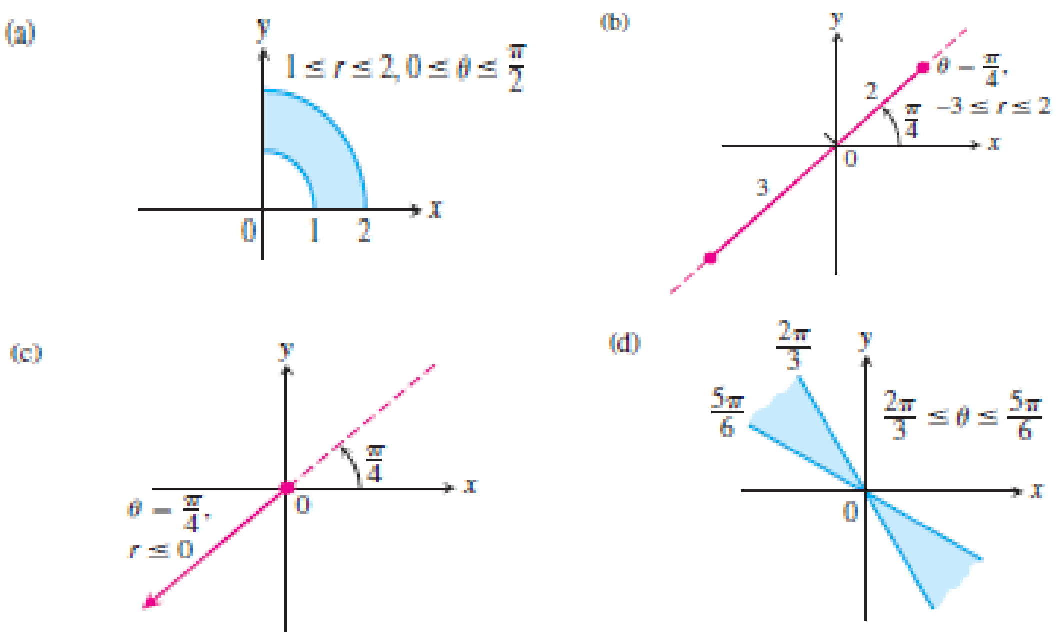
(a) $1 \le r \le 2$ and $0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \dfrac{\pi}{4}$

(c) $r \le 0$ and $\theta = \dfrac{\pi}{4}$

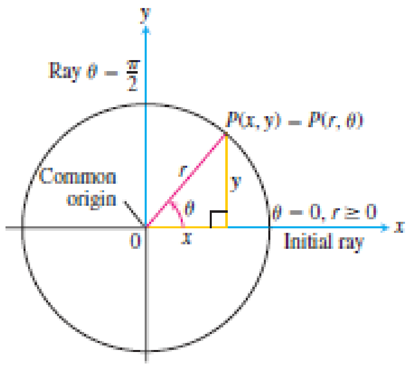
(b) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

Solution: The graphs are shown in Figures



Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive *x*-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive *y*-axis Figure.



The two coordinate systems are then related by the following equations.

Equation Relating Polar and Cartesian Coordinates

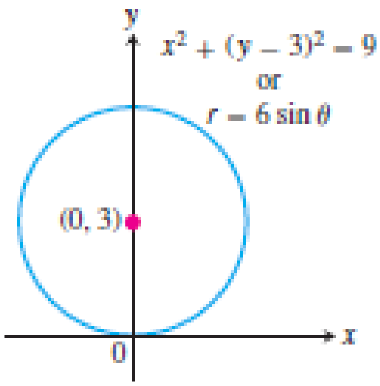
$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2$$

The first two of these equations uniquely determine the Cartesian coordinates *x* and *y* given the polar coordinates *r* and θ. On the other hand, if *x* and *y* are given, the third equation gives two possible choices for *r* (a positive and a negative value). For each selection, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (*x*, *y*). The other polar coordinate representations for the point can be determined from these two, as in Example 1.

Ex(4) **:-** Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't.

Ex(5) **:-** Find a polar equation for the circle $x^2 - (y-3)^2 = 9$ Figure.



Solution: $x^2+y^2-6y+9=9$

$$x^2+y^2-6y=0 \Rightarrow r^2-6r\sin\theta=0$$

$$r=0 \ \ or \ \ r-6\sin\theta=0 \Rightarrow r=6\sin\theta$$

Ex(6) **:-** Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

(a) $r\cos\theta=-4$

(b) $r^2=4r\cos\theta$

(c) $r=\dfrac{4}{2\cos\theta-\sin\theta}$

Solution: We use the substitutions $r\cos\theta=x, r\sin\theta=y,\ r^2=x^2+y^2$.

(a) $r\cos\theta=-4$

The Cartesian equation: $r\cos\theta=-4$

$$x=-4$$

The graph: Vertical line through $x=-4$ on the x-axis

(b) $r^2=4r\cos\theta$

The Cartesian equation: $r^2=4r\cos\theta$

$$x^2+y^2.=4x$$

$$x^2-4x+y^2.=0$$

$$x^2-4x+4+y^2.=4$$

$$(x-2)^2+y^2=4$$

The graph: Circle, radius 2, center $(h,k)=(2{,}0)$

(c) $r=\dfrac{4}{2\cos\theta-\sin\theta}$

The Cartesian equation: $r(2\cos\theta-\sin\theta)=4$

$$2r\cos\theta-r\sin\theta=4$$

$$2x-y=4$$

$$y=2x-4$$

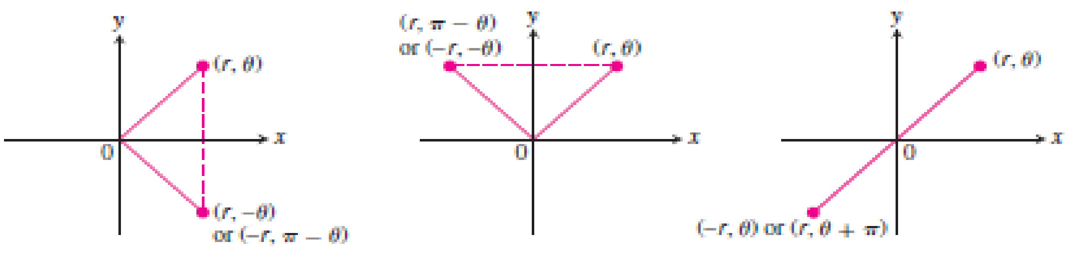
The graph: Line, slope $m=2,\ y$-intercept $b=-4$

Graphing in Polar coordinates

This section describes techniques for graphing equations in polar coordinates.

1.Symmetry

Illustrates the standard polar coordinate tests for symmetry.



(a) About the x-axis (b) About the y-axis (c) About the origin

Symmetry tests for Polar Graphs

a. Symmetry about the x-axis: If the point (r,θ) lies on the graph, the point $(r,-\theta)$ or $(-r,\pi-\theta)$ lies on the graph.

b. Symmetry about the y-axis: If the point (r,θ) lies on the graph, the point $(r,\pi-\theta)$ or $(-r,-\theta)$ lies on the graph.

c. Symmetry about the origin: If the point (r,θ) lies on the graph, the point $(-r,\theta)$ or $(r,\pi+\theta)$ lies on the graph.

2.Intercepts

1. To find the x-intercepts, put θ=0 and find the values of r.
2. To find the y-intercepts,put θ= or θ= ,and find the values of r.

Slope

The slope of a polar curve $r=f(\theta)$ is given by dy/dx, not by $r'=df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations $x=r\cos\theta=f(\theta)\cos\theta,\ \ y=r\sin\theta=f(\theta)\sin\theta.$

If f is a differentiable function of θ, then so are x and y ,and when $dx/d\theta\neq 0$, we can calculate dy/dx from the parametric formula

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\frac{d}{d\theta}\left(f(\theta)\cdot\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\cdot\cos\theta\right)}=\frac{\frac{d}{d\theta}\sin\theta+f(\theta)\cos\theta}{\frac{d}{d\theta}\cos\theta-f(\theta)\sin\theta}=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}$$

(Product rule for derivatives)

Slope of the curve

if $r=f(\theta)$ is differentiable and $dx/d\theta=0$, then the slope dy/dy at (r,θ) is given by the formula.

$$\text{slope at } (r,\theta)=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)}=\frac{f'(\theta)\sin\theta+f(\theta)\cos\theta}{f'(\theta)\cos\theta-f(\theta)\cdot\sin\theta},\quad provided\ dx/d\theta\neq 0\ at\,(r,\theta)$$

If the curve $r=f(\theta)$ passes through the origin at $\theta=\theta_0$, then $f(\theta_0)=0$ and the slope equation gives $\left.\frac{dy}{dx}\right|_{(0,\theta_0)}=\frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0}=\frac{r'\sin\theta_0}{r'\cos\theta_0}=\tan\theta_0.$

If the graph of $r=f(\theta)$ passes through the origin at the value $\theta=\theta_0$ the slope of the curve there is $\tan\theta_0$ The reason we say “slope at $(0,\theta_0)$ ” and not just “slope at the origin” is that a polar curve may pass through the origin (or any point) more than once, with different slopes at different θ-values. This is not the case in our first example, however.

Ex(1) **:-** Graph the curve $r=1-\cos\theta$

Solution: The curve is symmetric about the x-axis because

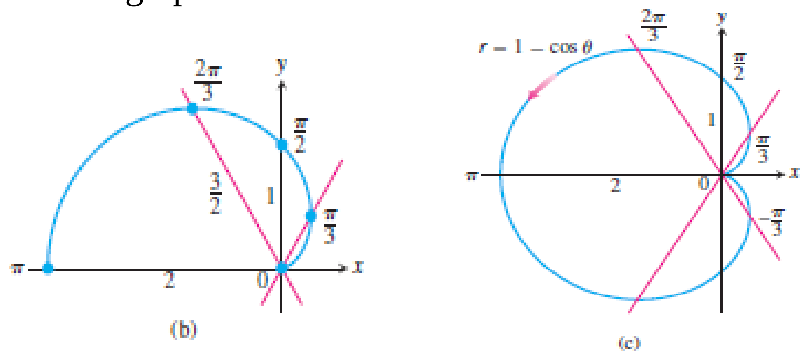
(r,θ) on the graph $\Rightarrow r=1-\cos\theta$

$$\Rightarrow r = 1 - \cos(-\theta)$$
$$\Rightarrow (r, -\theta) \text{ on the graph}$$

As θ increases from 0 to $\pi, \cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to $2\pi, \cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.
The curve leaves the origin with slope $\tan(0) = (0)$ and returns to the origin with slope $\tan(2\pi) = 0$.
We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the *x*-axis to complete the graph.

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



The curve is called a *cardioid* because of its heart shaped appearance. Cardioid shapes appear in the cams that direct the even layering of thread on bobbins and reels, and in the signal-strength pattern of certain radio antennas.
Ex(2) **:-** Graph the Curve $r^2 = 4\cos\theta$
Solution: The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the *x*-axis because

$$(r,\theta) \text{ on the graph} \Rightarrow r^2 = 4\cos\theta$$
$$\Rightarrow r^2 = 4\cos(-\theta) \qquad (\cos\theta = \cos(-\theta))$$
$$\Rightarrow (r, -\theta) \text{ on the graph}$$

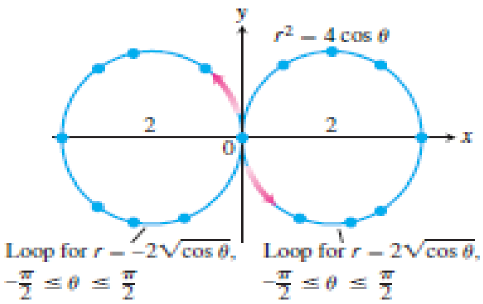
The curve is also symmetric about the origin because

$$(r,\theta) \text{ on the graph} \Rightarrow r^2 = 4\cos\theta$$
$$\Rightarrow (-r)^2 = 4\cos(\theta) \Rightarrow (-r,\theta) \text{ on the graph}$$

Together, these two symmetries imply symmetry about the *y*-axis.
The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of *r*: $r = \pm 2\sqrt{\cos\theta}$
We make a short table of values, plot the corresponding points, and use information about symmetry and tangents as aguide in connecting the points with a smooth curve.

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

A technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r,θ)-values, plot the corresponding points, and connect them in order of increasing θ. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. we can describe another method of graphing that is usually quicker and more reliable by using the following steps:

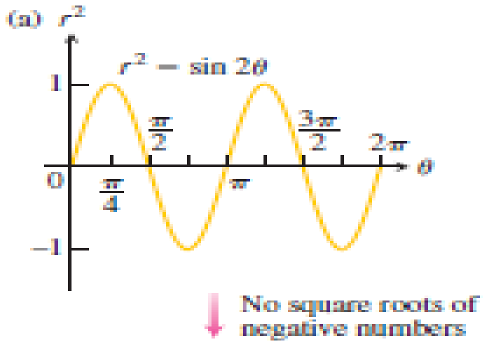
1. first graph $r = f(\theta)$ in the *Cartesian* $r\theta$-plan,(plot the values of θ on a horizontal axis and the corresponding values of r along a vertical axis).

2. Use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

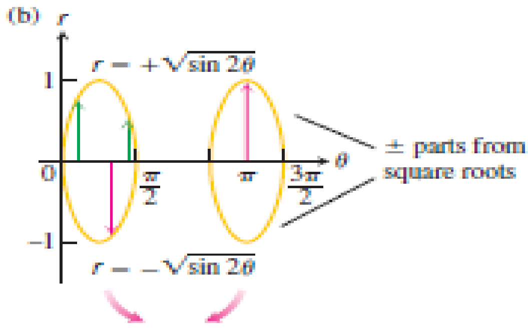
This method is better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where *r* is positive, negative, and nonexistent, as well as where *r* is increasing and decreasing. Here's an example.

Ex(3) **:-** graph the curve $r^2 = \sin 2\theta$

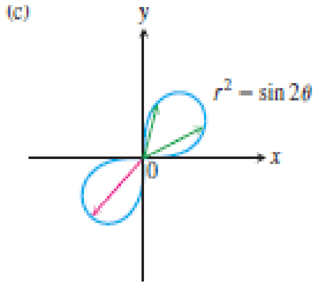
Solution: Here we begin by plotting r^2 (not *r*) as a function of θ in the Cartesian $r^2\theta$-plane. See Figure (a)



We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane, see Figure (b)



and then draw the polar graph see Figure (c).



The graph in Figure (b) "covers" the final polar graph in Figure (c) twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we actually learn a little more about the behavior of the function this way.

Finding points where polar graphs intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points at which polar graphs intersect. The problem is that a point of intersection may appear in the equation of one curve with different then it has in the equation of another curve. Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. The only sure way to identify all the points of intersection is to graph the equations.

Ex(4) **:-** Show that the point $(2, \pi/2)$ lies on the curve $r = 2\cos 2\theta$

Solution: It may seem at first that the point $(2, \pi/2)$ does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2\cos 2\left(\frac{\pi}{2}\right) = 2\cos \pi = -2,$$

Which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the same given point in which *r* is negative, for example, $(-2, -(\pi/2))$ when we try these in the equation $r = 2\cos 2\theta$, we find $-2 = 2\cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$ and the equation is satisfied. The point $(2, \pi/2)$ does lie on the curve after all.

Ex(5) **:-** Find the points of intersection of the curves $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$

Solution: In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others must be found by graphing.

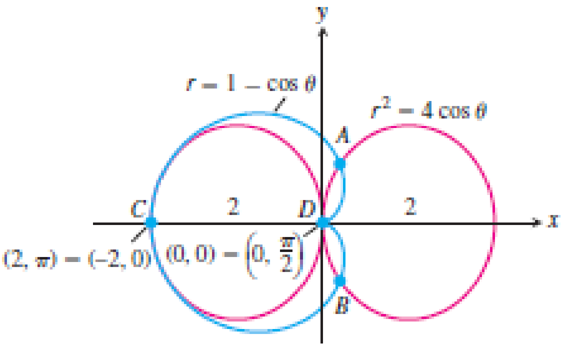
If we substitute $\cos\theta = r^2/4$ in the equation $r = 1 - \cos\theta$, we get

$$r = 1 - \cos\theta = 1 - \frac{r^2}{4}$$
$$4r = 4 - r^2$$
$$r^2 + 4r - 4 = 0$$
$$r = -2 \pm 2\sqrt{2} \qquad (Quadratic\ formula)$$

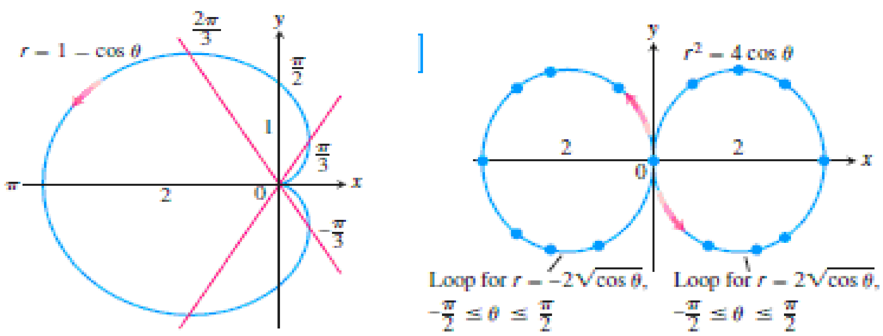
The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to give apoint on the curve. The values of θ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\theta = \cos^{-1}(1 - r) \qquad from\ r = 1 - \cos\theta$$
$$= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) \qquad set\ r = 2\sqrt{2} - 2$$
$$= \cos^{-1}\left(3 - 2\sqrt{2}\right) = \pm 80^0 \qquad (with\ a\ calculator\ rounded)$$

We have thus identified two intersection points: $(r, \theta) = \left(2\sqrt{2} - 2, \pm 80^0\right)$.
If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together Figure,



As we can now do easily by combining the graphs in Figures,



we see that the curves also intersect at the point $(2, \pi)$ and at the origin. Why weren't the r-values of these points revealed by the simultaneous solution? The answer is that the points (0, 0) and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of θ on each curves. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$ where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$.