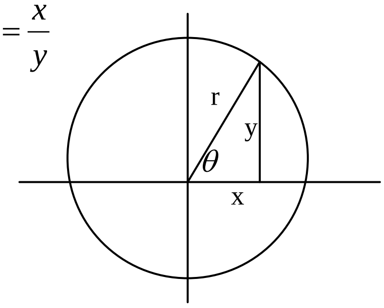
CHAPTER Six
Transcendental Functions

A function that is not algebraic is called transcendental. The class of transcendental functions includes the trigonometric, logarithmic, exponential and inverse trigonometric functions, and many more which are less familiar.

2.1 Trigonometric functions

When an angle of measure θ is placed in standard position at the center of a circle of radius r. the trigonometric functions of θ are defined by the equation

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, $\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$



$$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta} \qquad (1)$$

Since by the theorem of Pythagoras, we have $x^2 + y^2 = r^2$

It follows that $\sin^2\theta + \cos^2\theta = 1$

It is also useful to express the coordinates of P in term of r and θ as follows:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

When $\theta = 0$, we have y=0 and x=r

hence, from the definitions (1), we obtain $\sin\theta = 0, \quad \cos\theta = 0$

Similarly, for aright angle $\theta = \frac{\pi}{2}$, we have x=0. y=r

hence, $\sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0$

We include here a short table of the angles most frequently used, their radian measures, and their sines and cosines.

Degrees	0°	30°	45°	60°	90°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1

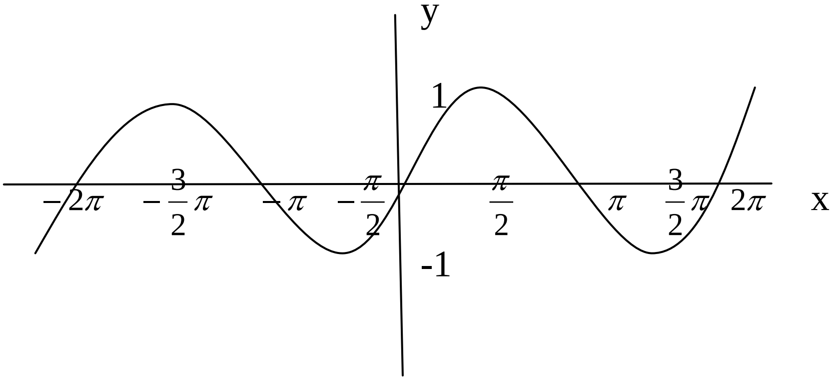
Graphs of trigonometric functions

1. $y=\sin x$

$D_{\sin} = R$

$R_{\sin} = [-1,1]$

x	Y		x	y
0	0		$-\frac{\pi}{2}$	-1
$\frac{\pi}{2}$	1		$-\pi$	0
π	0		$-\frac{3}{2}\pi$	1
$\frac{3}{2}\pi$	-1		-2π	0
2π	0			

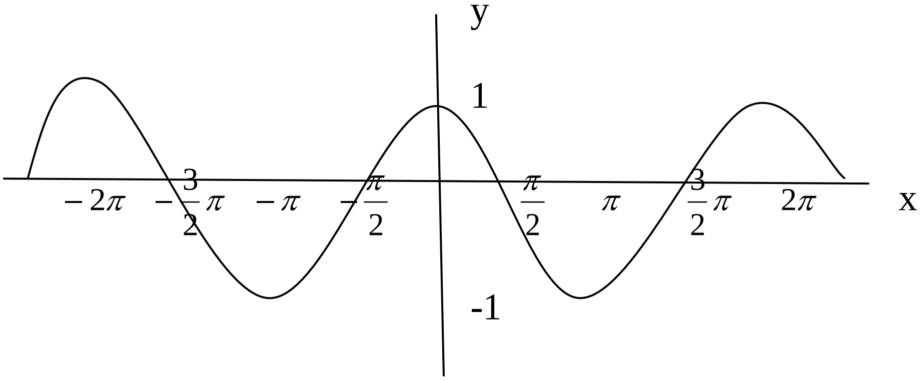


2. $y = \cos x$

$D_{\cos} = R$

$R_{\cos} = [-1,1]$

x	Y		x	y
0	1		$-\frac{\pi}{2}$	0
$\frac{\pi}{2}$	0		$-\pi$	-1
π	-1		$-\frac{3}{2}\pi$	0
$\frac{3}{2}\pi$	0		-2π	1
2π	1			



3. $y = \tan x$

$$y = \frac{\sin x}{\cos x}$$

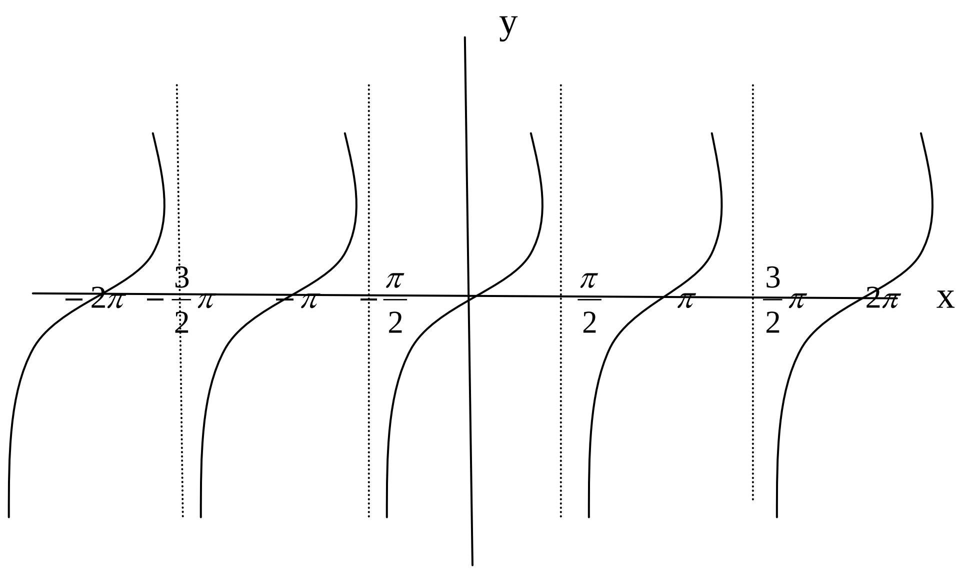
x	Y		x	y
0	0		$-\frac{\pi}{2}$	$-\infty$
$\frac{\pi}{2}$	∞		$-\pi$	0
π	0		$-\frac{3}{2}\pi$	∞
$\frac{3}{2}\pi$	∞		-2π	0
2π	0			

$$\cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, ...$$

$$x = \frac{\pm(2n+1)\pi}{2},\ n = 0,1,2,...$$

$$D_{\tan} = R / \left\{ x : x = \frac{\pm(2n+1)\pi}{2},\ n = 0,1,2,... \right\}\ or\ R / \frac{n\pi}{2},\ n \text{ is odd integer}$$

$$R_{\tan} = R$$



4. $y = \cot x$

$$y = \frac{\cos x}{\sin x}$$

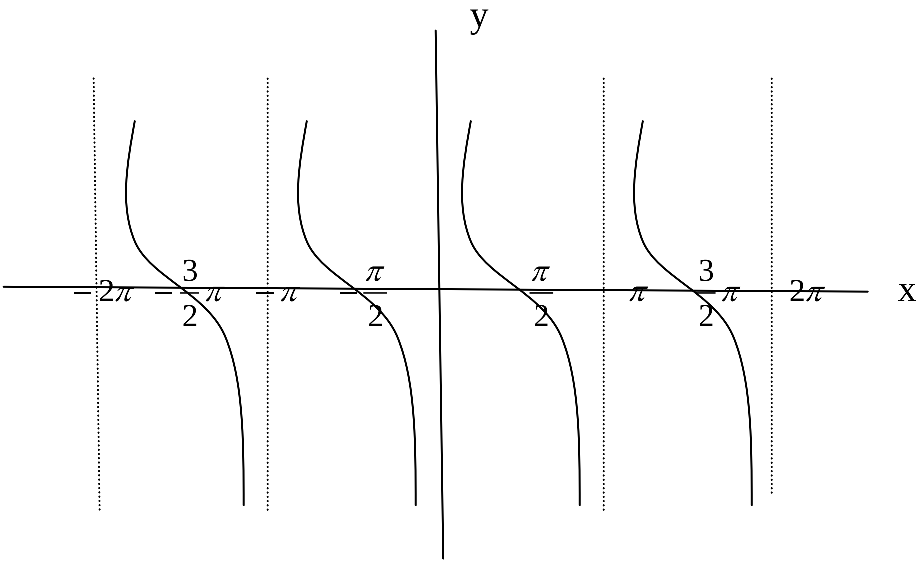
x	y		x	y
0	∞		$-\frac{\pi}{2}$	0
$\frac{\pi}{2}$	0		$-\pi$	∞
π	$-\infty$		$-\frac{3}{2}\pi$	0
$\frac{3}{2}\pi$	0		-2π	$-\infty$
2π	∞			

$\sin x = 0 \Rightarrow x = 0, \pm\pi, \pm 2\pi, ...$

$x = n\pi,\ n = 0, \pm1, \pm2, ...$

$D_{\cot} = R / \{x : x = n\pi,\ n = 0, \pm1, \pm2, ...\}$ n is integer

$R_{\cot} = R$



5. $y = \sec x$

$$y = \frac{1}{\cos x}$$

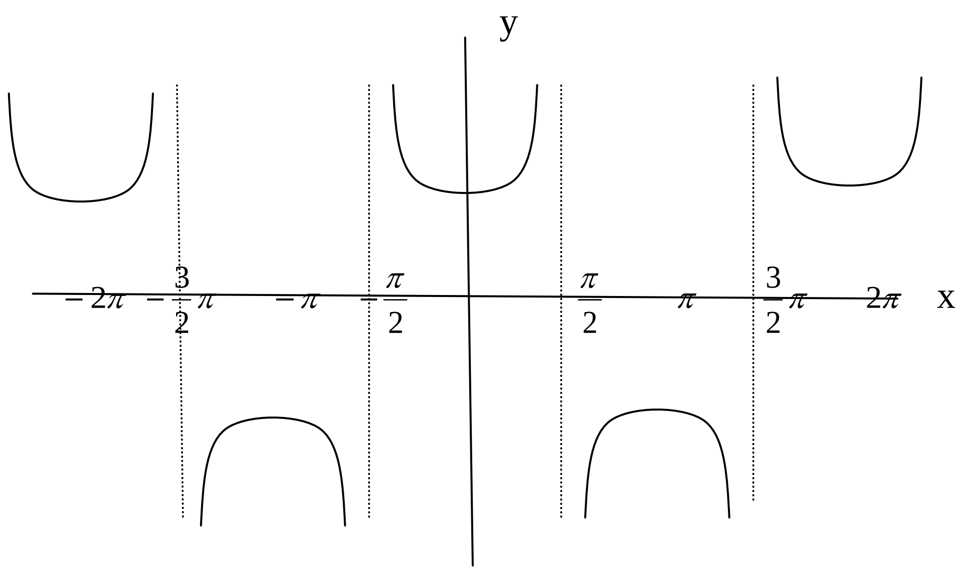
x	y		x	y
0	1		$-\frac{\pi}{2}$	∞
$\frac{\pi}{2}$	∞		$-\pi$	-1
π	-1		$-\frac{3}{2}\pi$	∞
$\frac{3}{2}\pi$	∞		-2π	1
2π	1			

$\cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, ...$

$x = \frac{\pm(2n+1)\pi}{2},\ n = 0,1,2,...$

$D_{\tan} = R / \left\{ x : x = \frac{\pm(2n+1)\pi}{2},\ n = 0,1,2,... \right\}$ *or* $R / \frac{n\pi}{2},\ n$ *is odd* int *eger*

$R_{\tan} = R/(-1,1)$



6. $y = \csc x$

$y = \frac{1}{\sin x}$

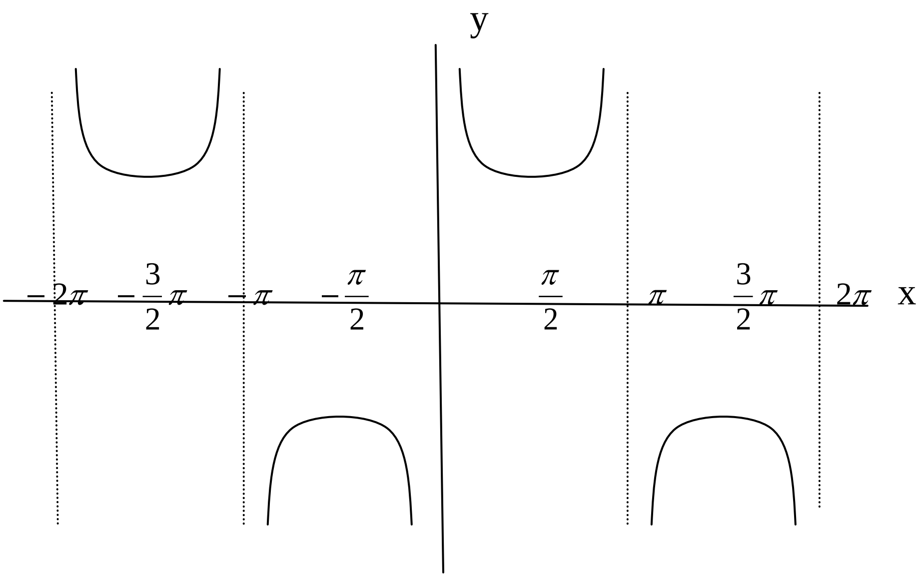
x	y		x	y
0	∞		$-\frac{\pi}{2}$	1
$\frac{\pi}{2}$	1		$-\pi$	∞
π	$-\infty$		$-\frac{3}{2}\pi$	1
$\frac{3}{2}\pi$	-1		-2π	$-\infty$
2π	∞			

$\sin x = 0 \Rightarrow x = 0, \pm\pi, \pm 2\pi, ...$

$x = n\pi,\ n = 0,\pm1,\pm2,...$

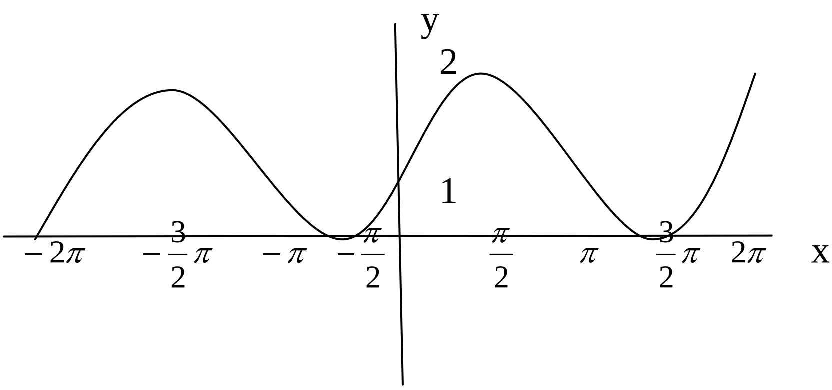
$D_{\csc} = R / \{x : x = n\pi,\ n = 0,\pm1,\pm2,...\}$ n *is* int *eger*

$R_{\csc} = R/(-1,1)$



Ex: 1. sketch the graph of the function $y = 1 + \sin x$ and find its domain and range
D = R , R = [0 , 2]

x	y		x	y
0	1		$-\frac{\pi}{2}$	0
$\frac{\pi}{2}$	2		$-\pi$	1
π	1		$-\frac{3}{2}\pi$	2
$\frac{3}{2}\pi$	0		-2π	1
2π	1			

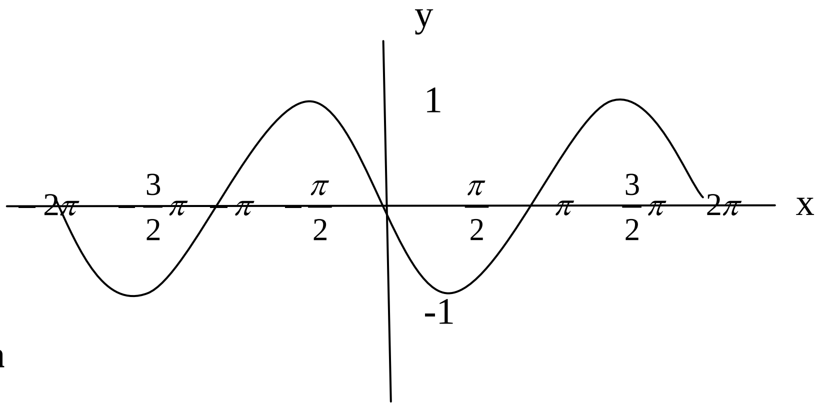


2. $y = \sin(-x)$

$-2\pi \le x \le 2\pi \Rightarrow 2\pi \ge -x \ge -2\pi$

$D = [-2\pi, 2\pi]$

$R = [-1, 1]$



2.2 Logarithmic Function

$y = \log_b x \Rightarrow x = b^y$

Ex: $y = \log_4 8$, find the value of y

$4^y = 8 \Rightarrow 2^{2y} = 2^3 \Rightarrow 2y = 3 \quad \therefore y = \frac{3}{2}$

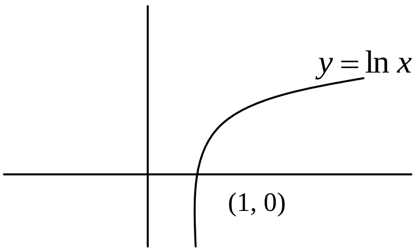
In particular:-

$x = e^y \Leftrightarrow y = \log x = \ln x$

$y = \ln x \Leftrightarrow x = e^y$**,** e is irrational number

$y = \ln x$ (Natural logarithm)

The graph of $y = \ln x$



$D = \{x : x > 0\}$

$R = (-\infty, \infty)$

Properties

For any a and c positive numbers

1. $\ln 1 = 0$
2. $\ln ac = \ln a + \ln c$
3. $\ln(a/c) = \ln a - \ln c$
4. $\ln a^r = r \ln a$
5. $\ln\left(\frac{1}{c}\right) = -\ln c$

Ex: find the following: $\ln 2 = 0.7$

a. $\ln 2^4 = 4 \ln 2 = 4(0.7) = 2.8$

b. $\ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2 = \frac{1}{2}(0.7) = 0.35$

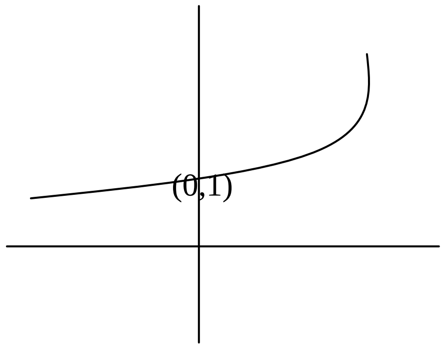
c. $\ln 8 = \ln 2^3 = 3\ln 2 = 3(0.7) = 2.1$

d. $\ln \frac{1}{2} = -\ln 2 = -0.7$

2.3 Exponential function

It is the inverse of logarithm function.

The natural exponential function:-



$x = \ln y \Leftrightarrow y = e^x$

$y = \exp(x)$

$D = R$

$R = \{y : y > 0\}$

$e^x = \ln^{-1} x$

$e^x : R \to (0, \infty)$

Properties

1. e is irrational number $\cong 2.718$
2. $e^0 = 1$
3. $\ln e = 1$
4. $e^x . e^y = e^{x+y}$
5. $e^x / e^y = e^{x-y}$
6. $e^{-x} = \frac{1}{e^x}$
7. $\ln e^x = x, \quad e^{\ln x} = x$

Ex: find the following

1. $e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$
2. $\ln\left(e^{-x^2}\right) = -x^2$
3. $e^{2\ln x} = e^{\ln x^2} = x^2$
4. $e^{x+\ln x} = e^x . e^{\ln x} = xe^x$
5. $e^{-\ln\left(\frac{1}{x}\right)} = \frac{1}{e^{\ln\frac{1}{x}}} = \frac{1}{\frac{1}{x}} = x$
6. $\ln\frac{1}{e^x} = \ln 1 - \ln e^x = 0 - x\ln e = 0 - x = -x$

Ex: prove that $2\ln\sin\theta = \ln(1-\cos\theta) + \ln(1+\cos\theta)$

$L.H = \ln(1-\cos\theta) + \ln(1+\cos\theta)$

$= \ln[(1-\cos\theta).(1+\cos\theta)]$

$= \ln(1-\cos^2\theta)$

$= \ln\sin^2\theta = 2\ln\sin\theta$

2.4 The function a^u and $\text{Log}_a u$

If a is any positive real number and

$b = \ln a$ ....(1)

Then $a = e^b$ ....(2)

This two equations may be combined to give the result $a = e^{\ln a}$, $\ln a$ is the exponent to which the base e must be raised to give a.

Def.1 let a be a positive number, then for any real number u, we define a^u to be $\exp(u \ln a)$

$$a^u = \exp(u \ln a)$$

Or $$a^u = e^{u \ln a}$$

Def.2 let a be a positive number not equal to one. The logarithmic function, to the base a, is defined on the domain x>0, by equation

$$u = \log_a x = \frac{\ln x}{\ln a}$$

Two useful operating rules

1. To remove exponentials, take the logarithm of both sides.

2. To remove logarithms from an equation, exponentiate both sides.

For all positive x, $e^{\ln x} = x$

Foe all x, $\ln(e^x) = x$

Ex:- Find m if $e^{2m} = 10$

Take the logarithm of both sides

$$e^{2m} = 10$$

$$\ln e^{2m} = \ln 10$$

$$2m = \ln 10$$

$$m = \frac{1}{2} \ln 10$$

Ex:- Find y if $\ln y = 3t + 5$

Exponentiate

$$e^{\ln y} = e^{3t+5}$$

$$y = e^{3t+5}$$

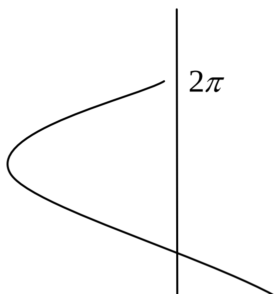
The inverse of trigonometric functions

Graphs of the inverse of trigonometric functions

$y = \sin^{-1} x$

$D = [-1,1]$

$R = R$



$\frac{3}{2}\pi$

π

$\frac{\pi}{2}$

-1 1

$-\frac{\pi}{2}$

$-\pi$

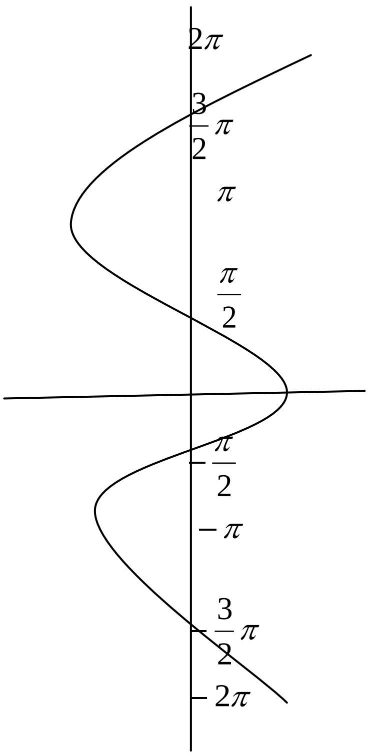
$-\frac{3}{2}\pi$

-2π

1. $y = \cos^{-1} x$

$D = [-1,1]$

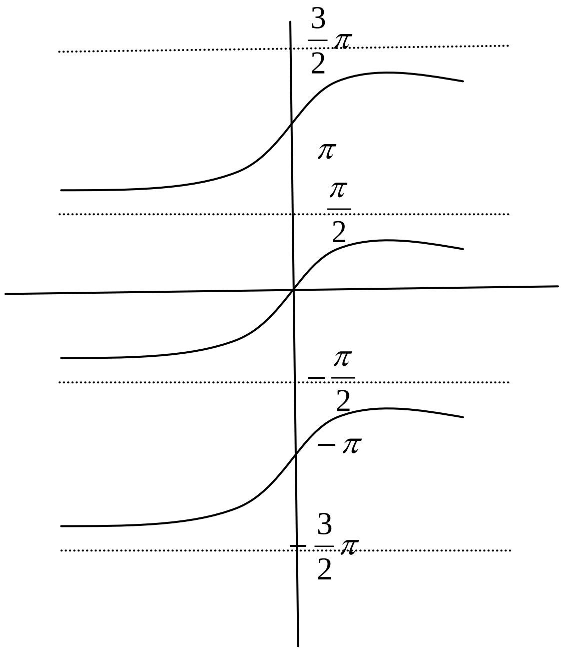
$R = R$



2. $y = \tan^{-1} x$

$D = R$

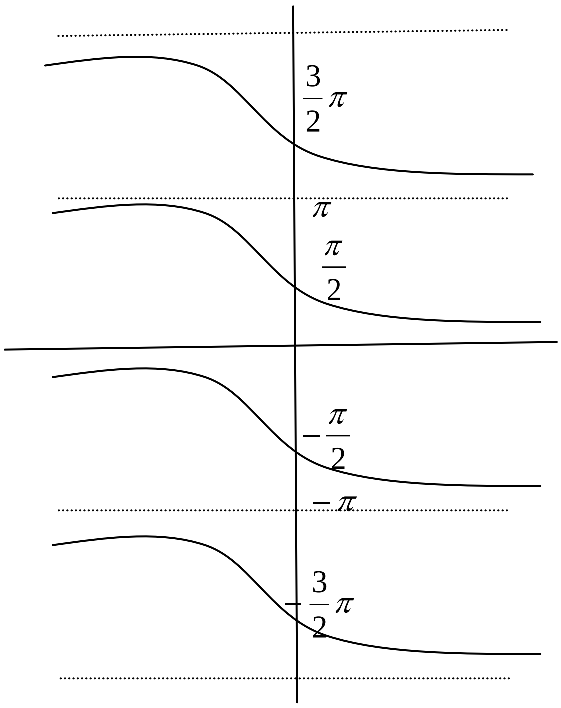
$R = R / \frac{n\pi}{2}, \; n \; is \; odd \; \text{int} \; eger$



3. $y = \cot^{-1} x$

$D = R$

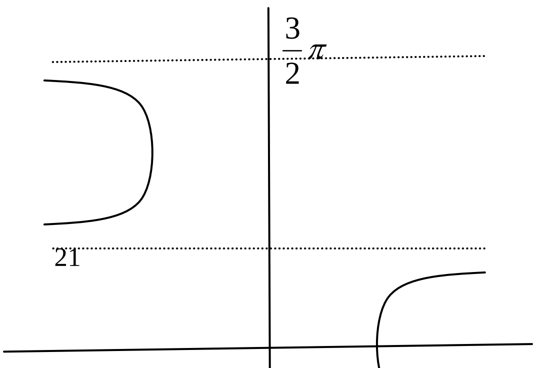
$R = R / \{n\pi, \ n \text{ is integer}\}$



4. $y = \sec^{-1} x$

$D = R / (-1, 1)$

$R = R / \left\{ \frac{n\pi}{2}, \ n \text{ is odd integer} \right\}$



π

$\frac{\pi}{2}$

$-\frac{\pi}{2}$

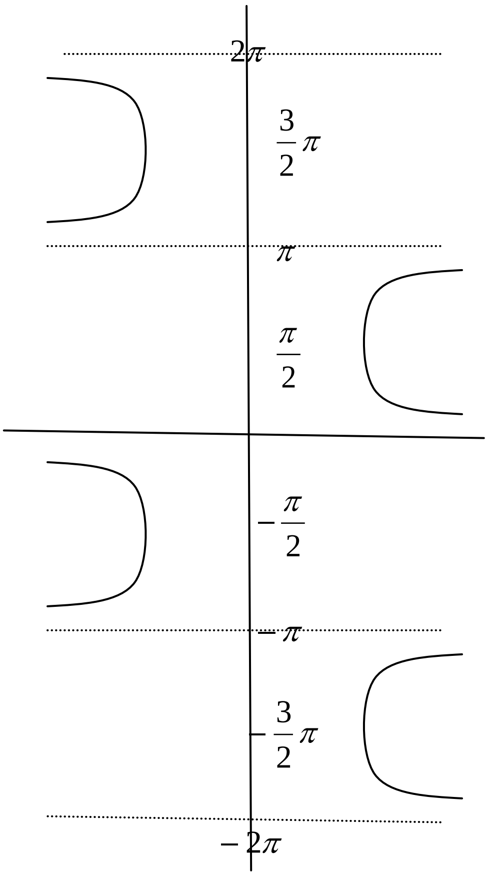
$-\pi$

$-\frac{3}{2}\pi$

5. $y = \csc^{-1} x$

$D = R/(-1,1)$

$R = R/\{n\pi,\ \ n\, is\, \mathrm{int}\, eger\}$



Exercises

1. sketch the graph of the following functions, then find its domain and range

a. $y = 1 + cocx$

b. $y = \cos(-x)$

c. $y = \tan^{-1}(x) + 1$

d. $y = 1 + \sec^{-1}(x)$

2. Let $y = \log_5 15$, find the value of y.

3. Let $y^2 = \log_2 4$, find the value of y.

4. Find the following $(\ln 5 = 1.6)$

1. $\ln 5^3$

2. $\ln 5^{\frac{1}{5}}$

3. $\ln 25$

4. $\ln \frac{1}{5}$

5. Find the following

a. $e^{-\ln\left(\frac{1}{\ln x}\right)}$

b. $e^{-\ln \sin x}$

c. $\ln\left(e^{-(x-1)}\right)$

d. $e^{\tan x + \ln x^2}$

6. Find m if $e^{4n^2} = 6$

7. Find y if $\ln y = 2x - t$