CHAPTER Five
Integration and Applications

Introduction

Def : A function y=F(x) is called an antiderivative of f(x) if the derivative of F is f .

Theorem(1): If F(x) is an antiderivative of f(x) then F(x)+c is also an antiderivative, where c is any constant.

Proof: Since F(x) is an antiderivative then according to define $\frac{dF(x)}{dx} = f(x)$

Now $\frac{d}{dx}[F(x)+c] = \frac{dF(x)}{dx} + \frac{dc}{dx} = f(x) + 0 = f(x)$

Hence F(x)+c is an antiderivative of f(x).

Indefinite Integrals

The set of all antiderivatives (or solutions) of f(x) is called the indefinite integral of f w.r.t x and denoted by $y = \int f(x)dx = F(x) + c$

Where the symbol $\int$ is called an "integral sign", the function f is called the integrand of the integral, c is called the constant of integration, and dx tell us that the variable of integration is x.

Given $\frac{dy}{dx} = f(x)$ find a function F(x) s.t $\frac{dF(x)}{dx} = f(x)$ then the Solution $y = \int f(x)dx = F(x) + c$

Some Integration Formulas

If u=u(x), then

1. $\int \frac{du}{dx} \cdot dx = u(x) + c$

2. $\int au(x)dx = a\int u(x)dx$, where a is constant .

3. $\int [u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x)]dx = \int u_1(x)dx + \int u_2(x)dx + \cdots + \int u_n(x)dx$

4. $\int u^n \frac{du}{dx} \cdot dx = \frac{u^{n+1}}{n+1} + c, n \neq -1$

Ex(1) **:-**Evaluate $I = \int \left(3x^4 - 2x^3 - x^{\frac{1}{2}} + 2x^{-2} + 5x^{-\frac{1}{3}} - \sqrt{2}\right)dx$

$$I = 3\int x^4 dx - 2\int x^3 dx - \int x^{\frac{1}{2}} dx + 2\int x^{-2} dx + 5\int x^{-\frac{1}{3}} dx - \sqrt{2}\int dx$$

$$= 3\frac{x^5}{5} - 2\frac{x^4}{4} - \frac{x^{3/2}}{3/2} + 2\frac{x^{-1}}{-1} + 5\frac{x^{2/3}}{2/3} - \sqrt{2}x + c$$

Ex(2) **:-** Find $I = \int (x^2 + 1)^2 2x dx$

$I = \frac{(x^2+1)^3}{3} + c$

Ex(3) :- $I = \int \sqrt[5]{3x-1}\,dx$

$I = \frac{1}{3}\int (3x-1)^{1/5} \cdot 3dx = \frac{1}{3}\frac{(3x-1)^{6/5}}{6/5} + c$

Ex(4) :- $I = \int \sec x(\sec x + \tan x)dx$

$= \int (\sec^2 x + \sec x \tan x)dx = \int \sec^2 xdx + \int \sec x \tan xdx$

$= \tan x + \sec x + c$

Ex(5) :- $I = \int \frac{dx}{x\sqrt{4x^2-1}}$

Let $u = 2x \quad du = 2dx$

$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}|u| + c = \frac{1}{2}\sec^{-1} 2x + c$

Ex(6) :- $I = \int \frac{\cos\phi}{1+\sin\phi}d\phi$

Let $u = 1 + \sin\phi \quad du = \cos\phi d\phi$

$= \int \frac{du}{u} = \ln|u| + c = \ln|1+\sin\phi| + c$

Ex(7) :- $I = \int \frac{xdx}{x^2 \ln 5}$

Let $u = x^2 \quad du = 2xdx$

$= \frac{1}{2}\int \frac{du}{u \ln 5} = \frac{1}{2}\log_5 x^2 + c$

Ex(8) :- $I = \int e^{x^3} x^2 dx$

Let $u = e^{x^3} \quad du = e^{x^3} 3x^2 dx$

$= \frac{1}{3}\int du = \frac{1}{3}u + c = \frac{1}{3}e^{x^3} + c$

Ex(9) :- $I = \int (\cos x)4^{-\sin x} dx$

Let $u = -\sin x \quad du = -\cos xdx$

$= -\int 4^u du = \frac{-4^u}{\ln 4} + c = \frac{-4^{-\sin x}}{\ln 4} + c$

Formulae

1. $\int u^n du = \frac{u^{n+1}}{n+1} + c, n \neq -1$

2. $\int \frac{du}{u} = \ln u + c$

3. $\int e^u du = e^u + c, e = 2.7183$

4. $\int a^u du = \frac{a^u}{\ln a} + c, a > 0$

5. $\int \sin udu = -\cos u + c$

6. $\int \cos udu = \sin u + c$

7. $\int \sec^2 udu = \tan u + c$

8. $\int \csc^2 u\,du = -\cot u + c$

9. $\int \sec u \tan u\,du = \sec u + c$

10. $\int \csc u \cot u\,du = -\csc u + c$

11. $\int \frac{du}{u\ln a} = \log_u a + c$

12. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$

13. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$

14. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c$

Definite Integrals

The integral $\int_a^b f(x)dx$ is called the definite integral of f(x) over interval [a,b]. Later, we shall show this integral is a number defined in a certain way as a limit of approximating sums over the interval from a to b on the x-axis.

Properties of Definite Integrals

If f(x) is continuous function on [a , b] . then

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, k is constant

4. $\int_a^b [f_1(x) + f_2(x) + \cdots + f_n(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_n(x)dx$

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a \le c \le b$

The Fundamental Theorem Integral Calculus

If f(x) is continuous function on [a,b] , and F(x) is any antiderivative of f(x) over [a,b] . then

$$\int_a^a f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Ex(10) **:-**Evaluate $I = \int_{-3}^{2} (6 - x - x^2)dx$

$$I = \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-3}^{2} = \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right) = \frac{125}{6}$$

Ex(11) **:-**Find $I = \int_{0}^{\pi} \sin x dx$

$$I = -\cos x\Big|_0^{\pi} = -\left(\cos\pi - \cos 0\right) = -(-1-1) = 2$$

Ex(12) **:-**Find $I = \int_{0}^{\pi/6} \frac{\sin 2x}{\cos^2 2x} dx$

$$I = \frac{-1}{2}\int_0^{\pi/6} \cos^{-2} 2x(-2)\sin 2x dx = \frac{-1}{2}\frac{\left(\cos 2x\right)^{-1}}{-1}\Bigg|_0^{\pi/6} = \frac{1}{2}\frac{1}{\cos 2x}\Bigg|_0^{\pi/6} = \frac{1}{2}\left[\frac{1}{\cos\frac{\pi}{3}} - \frac{1}{\cos 0}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1/2} - \frac{1}{1}\right] = \frac{1}{2}\left(2-1\right) = \frac{1}{2}$$

Ex(13) **:-**Show that if f(x) is continuous, then $\int_0^1 f(x)dx = \int_0^1 f(1-t)dt$

let $x = 1-t \Rightarrow dx = -dt$

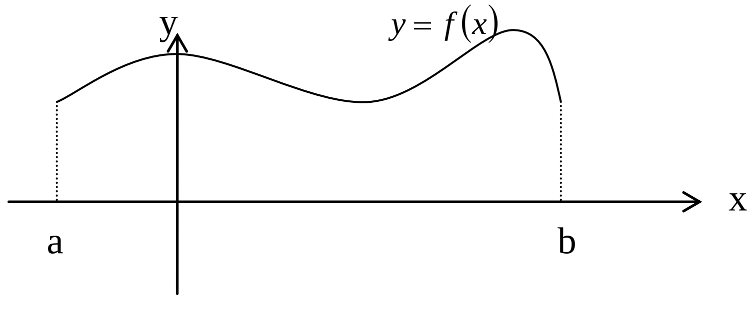
at $x = \begin{cases} 0 \Rightarrow t = 1 \\ 1 \Rightarrow t = 0 \end{cases}$

$$\int_0^1 f(x)dx = \int_1^0 f(1-t)\left(-dt\right) = -\int_1^0 f(1-t)dt = \int_0^1 f(1-t)dt$$

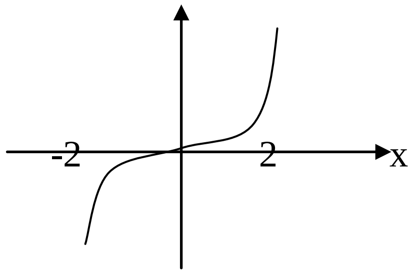
6.6 "Applications of Definite Integrals"

[1] The Area Under a Curve

If y=f(x) is nonnegative and integrable over a closed interval [a,b], then the integral of f from a and b is the area of the region between the graph of f and the x-axis from a to b. we some times calls this number the area under the curve y=f(x) from a to b.



Ex(1) **:-** find the areas of the regions enclosed by the carve $f(x)=4x^3$ and the lines x=-2, x=2.



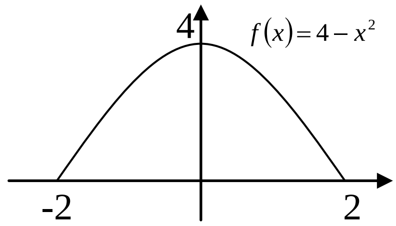
$$f(x)=4x^3 \Rightarrow 0=4x^3 \Rightarrow x=0\in[-2,2]$$

$$A=\left|\int_{-2}^{0}4x^3dx\right.+\left|\int_{0}^{2}4x^3dx=x^4\Big|_{-2}^{0}+x^4\Big|_{0}^{2}\right.$$

$$=|0-16|+|16-0|=32$$

Ex(2) **:-** find the area between the x-axis and the curve $f(x)=4-x^2$ for $-2\le x\le 2$.

Since $f(x)=4-x^2$ on [-2,2], then the area between the curve and x-axis from -2 to 2 is



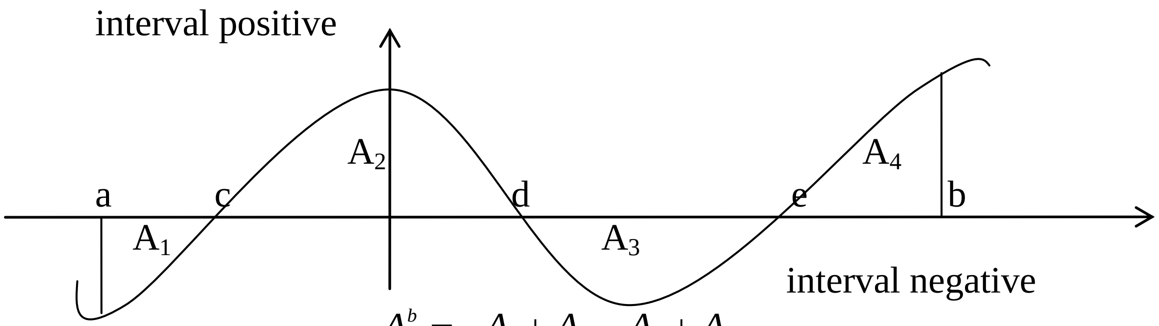
$$A=\int_{-2}^{2}\left(4-x^2\right)dx=4x-\frac{x^3}{3}\Big|_{-2}^{2}$$

$$=\left[4(2)-\frac{2^3}{3}\right]-\left[4(-2)-\frac{-2^3}{3}\right]$$

$$=\left[8-\frac{8}{3}\right]-\left[-8+\frac{8}{3}\right]=16-\frac{16}{3}=\frac{32}{3}$$

When the graph of y=f(x) crosses the x-axis between x=a and x=b. we find the area between the graph and the x-axis from a to b by taking the following steps:

1. find the points where f=0
2. use the zeros of f to partition [a,b] into subintervals
3. integrate f over each subinterval
4. add the absolute values of the results



$$A_a^b=-A_1+A_2-A_3+A_4$$

Then $A_a^b=\int_a^b f(x)dx=-\int_a^c f(x)dx+\int_c^d f(x)dx-\int_d^e f(x)dx+\int_e^b f(x)dx$

we illustrate these steps in the next example.

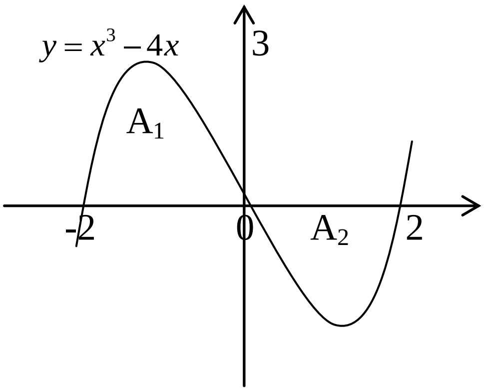
Ex(3) **:-** Find the area of region between the x-axis and the $y=x^3-4x, -2\le x\le 2$

1. the zero of y, we factor the formula for y to find where y is zero

$y=x^3-4x=x\left(x^2-4\right)=x(x-2)(x+2)$ the zeros occur at x=-2, 0 and 2.

2. the intervals of integration, the point x=-2, 0 and 2 partition [-2,2] into two subintervals [-2,0] and [0,2]

3. the integrations



$$\int_{-2}^{0}\left(x^3 - 4x\right)dx = \left|\frac{x^4}{4} - 2x^2\right|_{-2}^{0} = 0 - [4-8] = 4$$

$$\int_{0}^{2}\left(x^3 - 4x\right)dx = \left|\frac{x^4}{4} - 2x^2\right|_{0}^{2} = [4-8] - 0 = -4$$

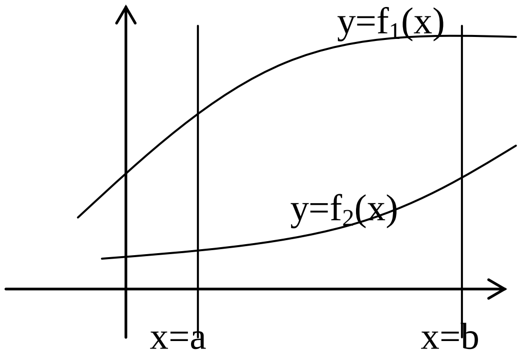
4. the absolute values added

Area of region $= |4| + |-4| = 8$

[2] Area Between Two Curves

Suppose that function $f_1(x)$ and $f_2(x)$ are continuous and that $f_1(x) \geq f_2(x)$ throughout an interval $a \leq x \leq b$, the area of region between the curves $y=f_1(x)$ and $y=f_2(x)$from a to b is the interval of (f_1-f_2) from a to b.

$$Area = \int_{a}^{b}\left(f_1(x) - f_2(x)\right)dx ....(2)$$



To apply eq. (2) we take the following steps:

1. graph the curves together. This tells you which is f_1 (upper curve) and which is f_2 (lower curve).
2. find the limits of integration.
3. write a formula for $f_1(x)-f_2(x)$. simplify it if you con.
4. integrate $f_1(x)-f_2(x)$ from a to b. then the number you get it is the area.

Ex(1) **:-** find the area between the curves $y = \cos x$ and $y = -\sin x$ from 0 to $\frac{\pi}{2}$.

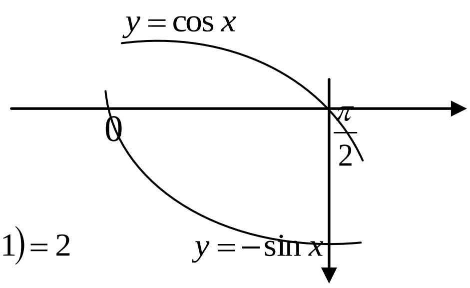
1. we graph the curves together the upper curve is $y = \cos x$ so we take $f_1(x) = \cos x$ in the area formula. The lower curve is $y = -\sin x$ so $f_2(x) = -\sin x$.

2. the limits of integration they are already given a=0 and $b = \frac{\pi}{2}$.

3. the formula for $f_1(x)-f_2(x)$ from step(1)

$$f_1(x) - f_2(x) = \cos x - (-\sin x) = \cos x + \sin x$$

4. integrate $f_1(x)-f_2(x)$ from a=0 to $b = \frac{\pi}{2}$



$$A = \int_{0}^{\frac{\pi}{2}}(\cos x + \sin x)dx = |\sin x - \cos x|_{0}^{\frac{\pi}{2}} = (1-0) - (0-1) = 2$$

Ex(2) **:-** find the area of region bounded by the curves $y = \frac{1}{4}x^2, y = x$.

Solu. $\frac{1}{4}x^2 = x \Rightarrow x^2 = 4x \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow x = 0 \quad and \quad x = 4$

$$A = \int_0^4 \left(\frac{1}{4}x^2 - x\right)dx = \left|\frac{x^3}{3} - \frac{x^2}{2}\right|_0^4 = \left(\frac{64}{12} - 8\right) - (0-0) = 32$$

Exercises: Find the areas of reigns enclosed by the lines and curves:-

1. The curves $y = 3x,\ y = x^2$ and the lines x=1, x=2.

2. The curves $y = x,\ y = x^3$.

3. The curves $y = \sqrt{x},\ y = x - 2$ and the x-axis.

4. The curves $y = x^2 - 2$ and the line y=2.

5. The x-axis and the curve $y = 2x - x^2$.

6. The curves $y = 2 - x^2$ and the line y=-x.

7. The curves $y = x^2$ and the line y=x.

8. The curves $y = x^2 - 2x$ and the line y=x.

9. The curves $x = y^2,\ x = -2y^2 + 3$.

10. The line $y = \cos\frac{\pi}{2}$ and the curve $y = 1 - x^2$.

11. Find the area of the triangular region bounded by y-axis and the curves $y = \sin x \quad and \quad y = \cos x$ in the first quadrant.

[2] Volume of a Solid of Revolution

Solid of revolutions are solids whose shapes can be generated by revolving plan regions about axes.

1. Volume of a solid of revolution (rotation about the x-axis).

The volume of the solid generated by revolving the region between the graph of a continuous function y=f(x) and the x-axis from x=a to x=b about the x-axis

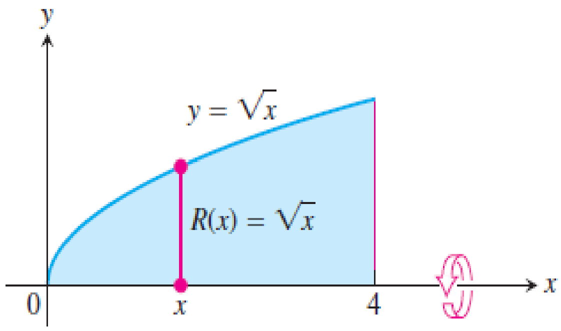
$$Volume = \int_a^b \pi(radius)^2\,dx = \int_a^b \pi(f(x))^2\,dx = \int_a^b \pi y^2\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

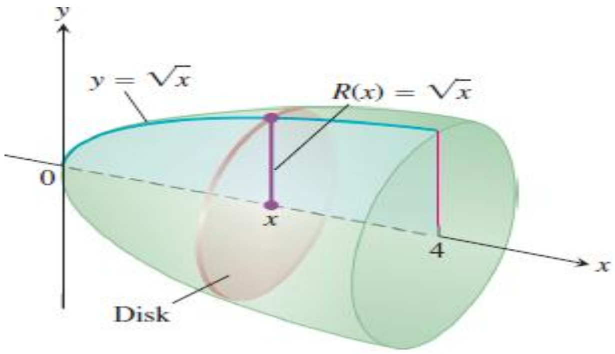
To apply equation.(1)

1. square the expression for the radius function f(y).
2. multiply by π
3. integrate from a to b.

Ex(1) :- the region between the curve $y = \sqrt{x}, 0 \le x \le 4$ and the x-axis is revolved about the x-axis. Find its volume.

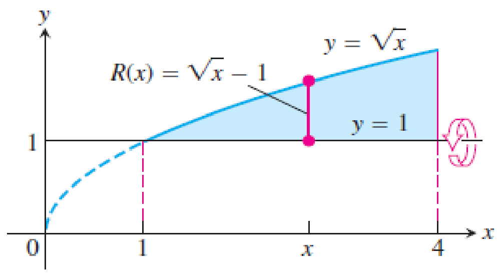
$$V = \int_a^b \pi(radius)^2\,dx = \int_0^4 \pi\left(\sqrt{x}\right)^2 dx = \int_0^4 \pi x\,dx = \pi \left.\frac{x^2}{2}\right|_0^4 = \pi\frac{(4)^2}{2} = 8\pi$$

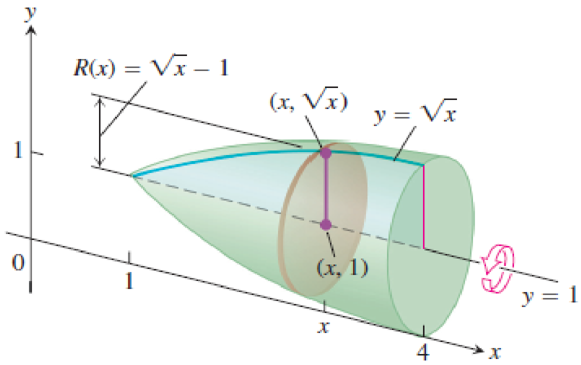




Ex(2) :- Find the volume generated by revolving the region bounded by $y=\sqrt{x}$ and the lines y=1 and x=4 about the line y=1.
The region runs from x=1 to x=4 at each x in the interval $1\le x\le 4$
radius at $x=\sqrt{x}-1$
therefore

$$V=\int_1^4 \pi(radius)^2\,dx=\int_1^4 \pi\left(\sqrt{x}-1\right)^2 dx=\pi\int_1^4\left(x-2\sqrt{x}+1\right)dx=\pi\left[\frac{x^2}{2}-2\cdot\frac{2}{3}x^{\frac{3}{2}}+x\right]_1^4=\frac{7\pi}{6}$$

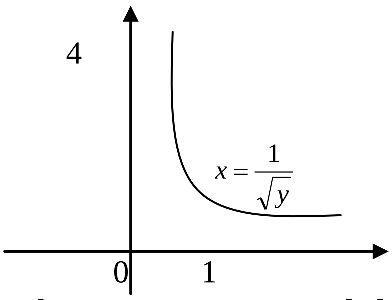




2. Volume of a solid of revolution (rotation about the y-axis).

$$Volume=\int_a^b \pi(radius)^2\,dy=\int_a^b \pi x^2\,dy$$

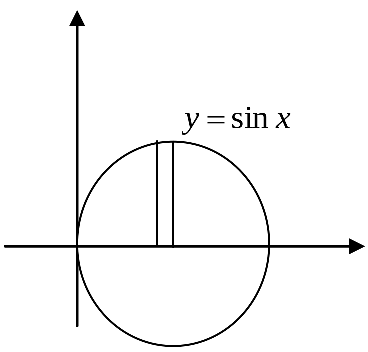
Ex(3) :- The region between the curve $x=\frac{1}{\sqrt{y}}, 1\le y\le 4$ is revolved about the y-axis to generate a solid. Find the volume of solid.



$$Volume=\int_1^4 \pi(radius)^2\,dy=\int_1^4 \pi\left(\frac{1}{\sqrt{y}}\right)^2 dy=\pi\int_1^4 \frac{1}{y}\,dy$$

$$=\pi\ln y\Big|_1^4=\pi\ln 4-0=\pi\ln 4=2\pi\ln 2$$

Ex(4) :- The region in the first quadrant bounded by the curve $y=\sin x$ and the x-axis. Find the volume of the resulting solid.



$$V=\int_0^\pi \pi y^2\,dx=\pi\int_0^\pi \sin^2 x\,dx=\pi\int_0^\pi\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$=\frac{\pi}{2}\left[\left(\pi-\frac{1}{2}\sin 2\pi\right)-\left(0-\frac{1}{2}\sin 0\right)\right]=\frac{\pi^2}{2}$$

π

Exercises:

1. Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines y=2 and x=0.
   a. about the x-axis.
   b. about the y-axis.
   c. about the line y=2.
   d. about the line x=4.

2. Find the volume of solid generated by the semicircle of radius r about the x-axis.

3. Find the volume of solid generated by the triangle with vertices (0,0), (h,0), (h,r) when it rotated about the x-axis.

Multiple Integration

7.1 Double Integrals

The symbol $\int_a^b\int_c^d f(x,y)dydx$ is called a double integral (or iterated integral) and is an abbreviation for $\int_a^b\left[\int_c^d f(x,y)dy\right]dx$

To evaluate a double integral you first compute the inner definite integral $\int_c^d f(x,y)dy$ taking the antiderivative of f with respect to y while keeping x fixed. The result will be a function of the single variable x, which you then integrate with respect to x between x=a and x=b.

Ex(1) **:-** Evaluate $\int_0^1\int_{-1}^2 xy^2dydx$

$$\int_{-1}^2 xy^2dy=\frac{1}{3}xy^3\Big|_{y=-1}^{y=2}=\frac{8}{3}x+\frac{1}{3}x=3x$$

$$\therefore\int_0^1\int_{-1}^2 xy^2dydx=\int_0^1 3xdx=\frac{3}{2}x^2\Big|_0^1=\frac{3}{2}$$

Ex(2) **:-** Evaluate $\int_0^1\int_{x^2}^{\sqrt{x}}160xy^3dydx$

$$\int_0^1\int_{x^2}^{\sqrt{x}} 160xy^3dydx = \int_0^1\left(40xy^4\Big|_{y=x^2}^{y=\sqrt{x}}\right)dx = \int_0^1\left[40x(\sqrt{x})^4 - 40x(x^2)^4\right]dx = \int_0^1[40x^3 - 40x^9]dx$$

$$= 10x^4 - 4x^{10}\Big|_0^1 = 6$$

Ex(3) **:-** Evaluate $\int_0^1\int_0^y y^2e^{xy}dxdy$

$$\int_0^1\int_0^y y^2e^{xy}dxdy = \int_0^1\left(ye^{xy}\Big|_{x=0}^{x=y}\right)dy = \int_0^1\left(ye^{y^2} - y\right)dy = \left(\frac{1}{2}e^{y^2} - \frac{1}{2}y^2\right)\Big|_0^1 = \left(\frac{1}{2}e - \frac{1}{2}\right) - \left(\frac{1}{2} - 0\right) = \frac{1}{2}e - 1$$

Ex(4) **:-** Evaluate $\int_0^1\int_{1-y}^1 e^y dxdy$

$$\int_0^1\int_{1-y}^1 e^y dxdy = \int_0^1\left(xe^y\Big|_{x=1-y}^{x=1}\right)dy = \int_0^1\left(e^y - (1-y)e^y\right)dy = \int_0^1 ye^y dy$$

$$= ye^y\Big|_0^1 - \int_0^1 e^y dy = \left(ye^y - e^y\right)\Big|_0^1 = 1$$

Exercises:

Evaluate the following double integrals

1. $\int_0^1\int_1^2 x^2 ydxdy$
2. $\int_1^2\int_0^1 x^2 ydydx$
3. $\int_0^{\ln 2}\int_{-1}^0 2xe^y dxdy$
4. $\int_2^3\int_{-1}^1 (x+2y)dydx$
5. $\int_0^1\int_{y-1}^{1-y} (2x+y)dxdy$
6. $\int_0^1\int_{x^2}^x 2xydydx$
7. $\int_0^1\int_0^x \frac{2y}{x^3+1}dydx$
8. $\int_0^1\int_y^1 ye^{x+y}dxdy$