CHAPTER Three
Differentiation and Applications

The Derivatives

Def. let y=f(x) be a function then the derivative of f w.r.t x denoted by $f'(x)=\frac{dy}{dx}=y'$, is defined by the rule $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, h\neq 0$ where $h=\Delta x$

(Differentiation Rules)

Rule 1:If f(x)=k where k is constant then $f'(x)=0$

Ex(1) $y=\sqrt{12}\Rightarrow \frac{dy}{dx}=0$

Rule 2**:**If f(x)=x^n where n is integer then $f'(x)=nx^{n-1}$

Ex(2)

1. $f(x)=x^5 \Rightarrow f'(x)=5x^4$

2. $y=2x^{-6}\Rightarrow y'=-12x^{-7}=\frac{-12}{x^7}$

3. $y=\sqrt{x}=x^{\frac{1}{2}}\Rightarrow \frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

Rule 3**:** If f(x)=k u(x) where k is constant and u is differentiable function of x then $f'(x)=ku'(x)$ or $\frac{dy}{dx}=k\cdot\frac{du}{dx}$

Ex(3) $f(x)=-4x^3 \Rightarrow f'(x)=-4*3x^2=-12x^2$

Rule 4**:** If f(x)=f_1(x)+f_2(x)+…+f_n(x) where f_1(x)f_2(x)…f_n(x) are differentiable function of x then $f'(x)=f_1'(x)+f_2'(x)+\ldots+f_n'(x)$

Ex(4) $f(x)=4x^3-2x^2+x-5\Rightarrow f'(x)=-12x^2-4x+1$

Rule 5**:** If f(x)=u(x)v(x) where u(x) and v(x) are differentiable function of x then $f'(x)=u(x)\cdot v'(x)+v(x)\cdot u'(x)$

Ex(5)

$$y=(x^2+3x-2)(x^3-8x^2+1)$$

$$\frac{dy}{dx}=(x^2+3x-2)(3x^2-16x)+(x^3-8x^2+1)(2x+3)$$

Rule 6**:** If $f(x)=\frac{u(x)}{v(x)}, v(x)\neq 0$ where u(x) and v(x) are differentiable functions of x then $f'(x)=\frac{v(x)u'(x)-u(x)v'(x)}{[v(x)]^2}$

Ex(6) $y=\frac{2x^3-4x^2+8}{x^2+1}\Rightarrow \frac{dy}{dx}=\frac{(x^2+1)(6x^2-8x)-(2x^3-4x^2+8)(2x)}{(x^2+1)^2}$

Rule **7:** If $f(x)=[u(x)]^n$ where n is integer and u(x) is a differentiable function of x then $f'(x)=n[u(x)]^{n-1}\cdot u'(x)$

Ex(7)

1. $y=(2x^2+5x-3)^{-4} \Rightarrow \dfrac{dy}{dx}=-4(2x^2+5x-3)^{-5}(4x+5)$

2. $y=(3x^{\frac{1}{2}}+5)^{-\frac{1}{4}} \Rightarrow \dfrac{dy}{dx}=-\dfrac{1}{4}(3x^{\frac{1}{2}}+5)^{-\frac{5}{4}}(\dfrac{3}{2}x^{-\frac{1}{2}})$

Exercises:

Find $\dfrac{dy}{dx}$

1. $y=x^2-x$

2. $y=\dfrac{x^4}{2}-\dfrac{3}{2}x^2-x$

3. $y=(x-1)(x^2+x+1)$

4. $y=(3x-1)(2x+5)$

5. $y=(x^2+1)^6(x^3+2x-1)^5$

6. $y=(x^2+1)^{-2}(x^{-3}+2x)^3$

Implicit Differentiation

Consider the function defined by the equation f(x,y)=0 which may or may not be solved for y in terms of x. for example $y+x^3+2x-5=0$ can be written as $y=-(x^3+2x-5)$ and $\dfrac{dy}{dx}=3x^2+2$ while $y^5+4x^2y^2+x^3-2=0$ can not be solved for y in terms of x. Implicit differentiation enables us to find the derivative of such function whenever they exist.

Ex(8) $y^2=x \Rightarrow 2y\dfrac{dy}{dx}=1 \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{2y}$

Ex(9) find the tangent and normal of the curve $x^2-xy+y^2=7$ at point (-1,2).

$$x^2-xy+y^2=7$$

$$2x-[xy'+y\cdot 1]+2yy'=0$$

$$2x-xy'-y+2yy'=0$$

$$(2y-x)y'=y-2x$$

$$y'=\frac{y-2x}{2y-x}$$

$$y'\big|_{(-1,2)}=\left.\frac{y-2x}{2y-x}\right|_{(-1,2)}=\frac{2-2(-1)}{2(2)-(-1)}=\frac{4}{5}$$

The tangent of the curve at (-1,2) is $y-2=\frac{4}{5}\left(x-(-1)\right)\Rightarrow y=\frac{4}{5}x-\frac{14}{5}$

The normal of the curve at (-1,2) is $y-2=-\frac{5}{4}\left(x+1\right)\Rightarrow y=-\frac{5}{4}x-\frac{3}{4}$

Ex(10) $x^2-y^2=1 \Rightarrow 2x-2yy'=0\Rightarrow y'=\frac{2x}{2y}=\frac{x}{y}$

The Second and Higher Derivatives

Given the function y=f(x), the derivative $y'=f'(x)=\frac{dy}{dx}$ is the 1st derivative of y , w.r. to x. And $y''=f''(x)=\frac{d^2y}{dx^2}$ is called the 2nd derivative of y, w.r. to x.

Thus the 2nd derivative is derivative of the 1st derivative.

That $\frac{d^2y}{dx^2}=\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)$

In general if y=f(x) is differentiable function of x then the nth derivative of y , w.r to x is denoted by $y^n=f^n(x)=\frac{d^ny}{dx^n}$

Ex(11) If $y=3x^4-5x^3+6x-7$ find $\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3},\ldots,\frac{d^ny}{dx^n}$

$$\frac{dy}{dx}=12x^3-15x^2+6$$

$$\frac{d^2y}{dx^2}=36x^2-30x$$

$$\frac{d^3y}{dx^3}=72x-30$$

$$\frac{d^4y}{dx^4}=72\Rightarrow\frac{d^5y}{dx^5}=0=\frac{d^6y}{dx^6}=\cdots=\frac{d^ny}{dx^n}=0$$

Ex(12) If $y=(x^3+2x-1)^{\frac{1}{2}}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx}=\frac{1}{2}(x^3+2x-1)^{-\frac{1}{2}}(3x^2+2)$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left\{(x^3+2x-1)^{-\frac{1}{2}}(6x)+(3x^2+2)\left[-\frac{1}{2}(x^3+2x-1)^{-\frac{3}{2}}(3x^2+2)\right]\right\}$$

Chain Rule and Parametric Equations

If y is a function of x, say y=f(x)
And x is a function of t, say x=g(t)

Then y is a function of t and $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$ -------- (1)

Ex(13) let $y=x^3-2x^2+3$ and $x=t^2+2$ find $\frac{dy}{dt}$ at t=2

$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=(3x^2-4x)(2t)$

When t=2 $\Rightarrow$ x=4+2=6

$$\therefore \left.\frac{dy}{dt}\right|_{t=2}=\left[3(6)^2-4*6\right](2*2)=336$$

or

$y=x^3-2x^2+3,\ x=t^2+2$

$y=(t^2+2)^3-2(t^2+2)^2+3$

$$\frac{dy}{dt}=3(t^2+2)^2(2t)-4(t^2+2)(2t)$$

$$\left.\frac{dy}{dt}\right|_{t=2}=3(4+2)^2(2-2)-4(4+2)(2\cdot 2)=336$$

L'Hopital Rule

Suppose that f(a)=g(a)=o or ∞ and $f'(a)$, and $g'(a)$ exist with $g'(a)\neq 0$

Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}$

Ex(16) find the following limits

1. $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{2x}{1}=2*2=4$

2. $\lim_{x\to 0}\frac{x-2x^2}{3x^2+5x}=\lim_{x\to 0}\frac{1-4x}{6x+5}=\frac{1-0}{0+5}=\frac{1}{5}$

3. $\lim_{x\to\infty}\frac{6x+5}{3x-8}=\lim_{x\to\infty}\frac{6}{3}=\frac{6}{3}=2$

Ex(17)

1. $\lim_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1}=\lim_{x\to 1}\frac{3x^2-3}{3x^2-2x-1}=\lim_{x\to 1}\frac{6x}{6x-2}=\frac{6}{6-2}=\frac{3}{2}$

2. $\lim_{x\to 0}\frac{\sqrt{1+x}-\frac{x}{2}-1}{x^2}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}=\lim_{x\to 0}\frac{-\frac{1}{4}\frac{1}{(1+x)^{\frac{3}{2}}}}{2}=\frac{-\frac{1}{4}*1}{2}=-\frac{1}{8}$

1. Derivatives of the Trigonometric functions

Theorems:-

1. If y=f(x)=sinx then $\frac{dy}{dx} = f'(x) = \cos x$
2. If y=f(x)=cosx then $\frac{dy}{dx} = f'(x) = -\sin x$
3. If y=f(x)=tanx then $\frac{dy}{dx} = f'(x) = \sec^2 x$
4. If y=f(x)=cotx then $\frac{dy}{dx} = f'(x) = -\csc^2 x$
5. If y=f(x)=secx then $\frac{dy}{dx} = f'(x) = \sec x \tan x$
6. If y=f(x)=cscx then $\frac{dy}{dx} = f'(x) = -\csc x \cot x$

Now: If u=u(x) is a differentiable function of x and

1. y=sinu then $\frac{dy}{dx} = \cos u \cdot \frac{du}{dx}$

2. y=cosu then $\frac{dy}{dx} = -\sin u \cdot \frac{du}{dx}$

3. y=tanu then $\frac{dy}{dx} = \sec^2 u \cdot \frac{du}{dx}$

4. y=cotu then $\frac{dy}{dx} = -\csc^2 u \cdot \frac{du}{dx}$

5. y=secu then $\frac{dy}{dx} = \sec u \cdot \tan u \cdot \frac{du}{dx}$

6. y=cscu then $\frac{dy}{dx} = -\csc u \cdot \cot u \cdot \frac{du}{dx}$

Ex(1) find $\frac{dy}{dx}$ of the following

1. $y=\sin(x^3+3x^2-2) \Rightarrow \frac{dy}{dx}=\cos(x^3+3x^2-2)(3x^2+6x)=(3x^2+6x)\cos(x^3+3x^2-2)$

2. $y=\cos^2(x^2+8x-9) \Rightarrow \frac{dy}{dx}=2\cos(x^2+8x-9)[-\sin(x^2+8x-9)](2x+8)$

3. $y=\tan 2x \cos(x^2+1) \Rightarrow \frac{dy}{dx}=\tan 2x[-\sin(x^2+1)(2x)]+\cos(x^2+1)\sec^2 2x(2)$

4. $y=\tan^{-3}(3x^2+\sec^2 5x)$

$\frac{dy}{dx}=[-3\tan^{-4}(3x^2+\sec^2 5x)][\sec^2(3x^2+\sec^2 5x)][6x+2(\sec 5x)(\sec 5x)(\tan 5x)5]$

5. $y=\dfrac{\sec[\sin(x^2+x^{-2})]}{\tan(x^3+1)}$

$$\frac{dy}{dx}=\frac{[\tan(x^3+1)][\sec\{\sin(x^2+x^{-2})\}\cdot\tan\{\sin(x^2+x^{-2})\}][\cos(x^2+x^{-2})\cdot(2x-2x^{-3})]}{\tan^2(x^3+1)}$$
$$\frac{-\sec[\sin(x^2+x^{-2})]\cdot\sec^2(x^3+1)\cdot 3x^2}{}$$

Ex(2) find $\frac{dy}{dx}$ if $x^2+5x-\tan^2(xy)=0$

$$2x+5-2\tan(xy)\cdot\sec^2(xy)\cdot\left(x\frac{dy}{dx}+y(1)\right)=0$$

$$2x+5-2y\tan(xy)\cdot\sec^2(xy)=2x\tan(xy)\cdot\sec^2(xy)\cdot\frac{dy}{dx}$$

$$\therefore\frac{dy}{dx}=\frac{2x+5-2y\tan(xy)\cdot\sec^2(xy)}{2x\tan(xy)\cdot\sec^2(xy)}$$

Ex(3) find $\frac{dy}{dx}$ if $2y=x^2+\sin y$

$$2y'=2x+\cos yy'$$

$$2y'-\cos yy'=2x\Rightarrow y'(2-\cos y)=2x\quad\Rightarrow y'=\frac{2x}{2-\cos y}$$

Ex(4) If y=sect and x=csct find $\frac{dy}{dx}$ at t=$\frac{\pi}{4}$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sec t\cdot\tan t}{-\csc t\cdot\cot t}=\frac{\frac{1}{\cos t}\cdot\frac{\sin t}{\cos t}}{-\frac{1}{\sin}\cdot\frac{\cos t}{\sin t}}=-\frac{\sin^3 t}{\cos^3 t}=-\tan^3 t$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=-\tan^3(\frac{\pi}{4})=-(1)^3=-1$$

Ex(5) Evaluate the following limit

$$\lim_{x\to0}\frac{\sin x}{x}=\lim_{x\to0}\frac{\cos x}{1}=\cos 0=1$$

Exercises

1. Find $\frac{dy}{dx}$ when y equals the given expression:

a. $3x+x\tan x$

b. $\sin x\sec x$

c. $\dfrac{4}{\cos x}$

d. $2x+\cot x$

e. $\dfrac{\cot x}{1+\cot x}$

f. $\dfrac{x}{1+\cos x}$

2. Find $\frac{dy}{dx}$ when y equals the given expression:

a. $\sin^2(3x-2)$ **c.** $\cot(3x-1)$

b. $\left(1-\tan\left(x/2\right)\right)^{-2}$

3. Evaluate the following limits

a. $\lim_{x\to 0}\frac{\sin 2x}{2x^2+x}$

Applications of Derivatives

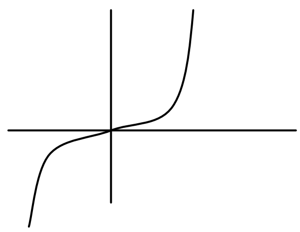
[1] Curve Sketching

Suppose that y=f(x) is a cont. function on interval I and has derivative at every point x$\in$I, then

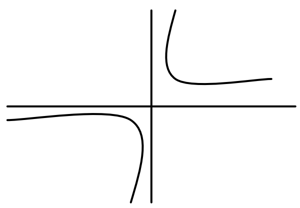
1. f(x) increases on I if $f'(x)>0, \forall x\in I$

2. f(x) decreases on I if $f'(x)<0, \forall x\in I$

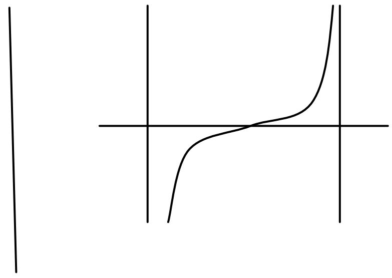
Ex(1) $y=x^3$ increases on $(-\infty,\infty)$



Ex(2) $y=\frac{1}{x}$ decreases on $(-\infty,0)$



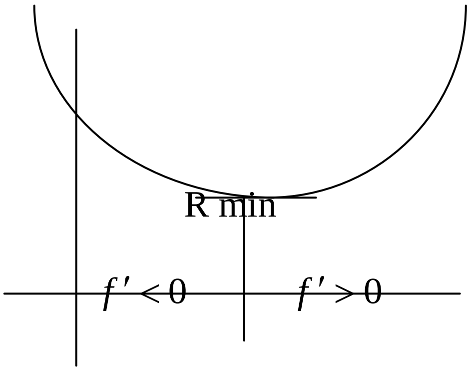
Ex(3) $y=\tan x$ increases on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

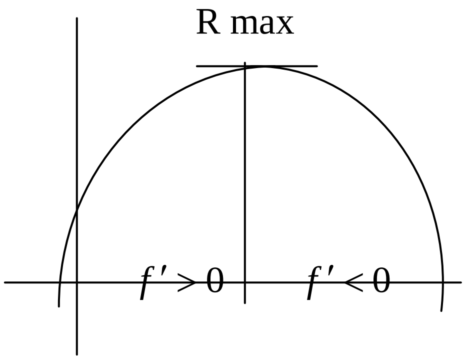


Def A critical point x of a function f(x) is the value of x where $f'(x_0)=0$

Suppose that f(x) is cont. at x_0

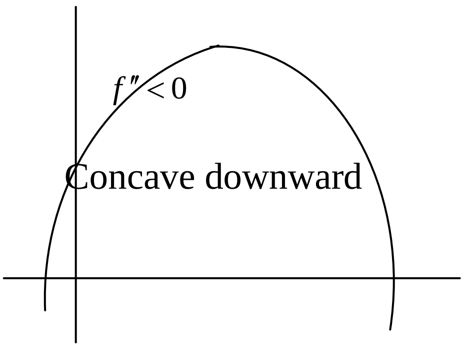
If f(x) is increasing (decreasing) in an interval I, with x_0 as aright endpoint and decreasing (increasing) in an interval I_2 with x_0 as left endpoint then f(x) has relative maximum (minimum) at x_0.

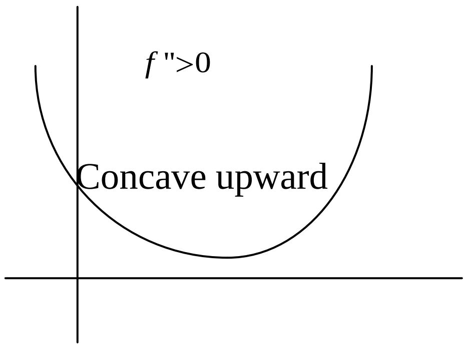




Ex $y=x^2$ decreases on $(-\infty,0)$ where the derivative $y'=2x$ is negative and increases on $(0,\infty)$ where the derivative is positive.

Def the graph of a differentiable function y=f(x) is concave up on an interval I where y' is increasing and concave down on an interval I where y' is decreasing.





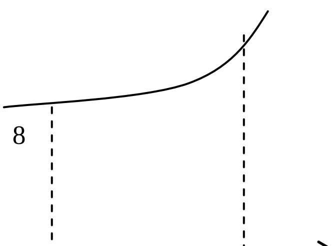
The second derivative test for concavity

The graph of y=f(x) is concave down on any interval I where $y''<0$ and concave up on any interval I where $y''>0$.

Def **:-** A point on the curve y=f(x) where the concavity changes from upward to downward (or vice versea) is called a point of inflection.

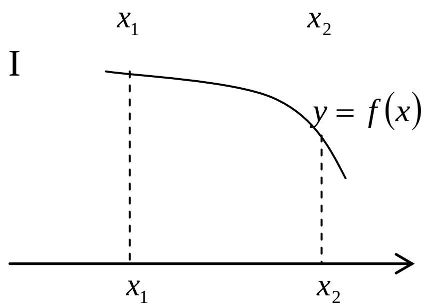
At a point of inflection on the graph of a twice differentiable function, $y''=0$

Def :-



1. if, for any two points x_1 and x_2 in I $y = f(x)$
$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

2. if, for any two points x_1 and x_2 in I
$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$



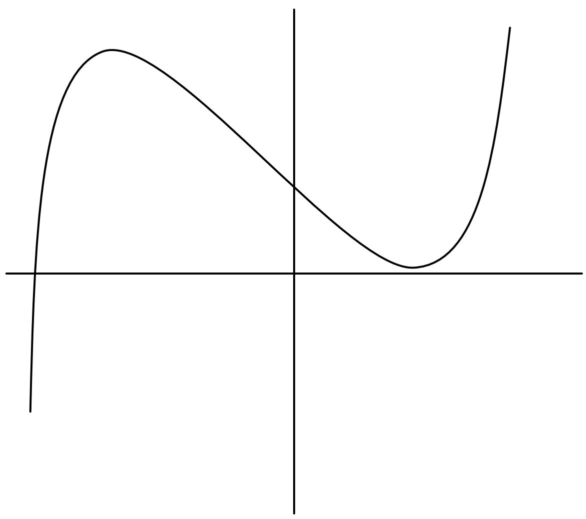
Steps in Graphing y=f(x)

1. Find y', y".
2. Find where y' is positive, negative, and zero. This will show where the curve may have local maxima and minima. And where the curve is rising and falling.
3. Find where y" is positive, negative, and zero. This will tell us about concavity and possible inflection points.
4. Make a summary table, we include the values of y, y' and y" at the intercepts and at the other important points, we summarize what we have learned about the curves behavior.
5. Draw the graph . in order to do this, we plot the points from the table and sketch the tangents at these points. Then we draw the curve by using the information about rise, fall, and concavity.

Ex:**-** Discuss and sketch the curve of the following functions:

a. $y=x^3-3x+2$

D: all x
R: all y



Interceptes:
Set x=0 $\Rightarrow$ y=2
$\therefore$ (0,2) is y-int.
Set y=0 $\Rightarrow$ $x^3-3x+2=0$
$(x-1)(x^2+x-2)=0$
$(x-1)^2(x+2)=0$
$\therefore$ x=1,-2
$\therefore$ (1,0),(-2,0) are x-int.
Symm**:** No
Asymptote**:** No

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$$

$$\frac{dy}{dx} = 0 \Rightarrow 3(x-1)(x+1) = 0$$

x=1,-1 are critical points .

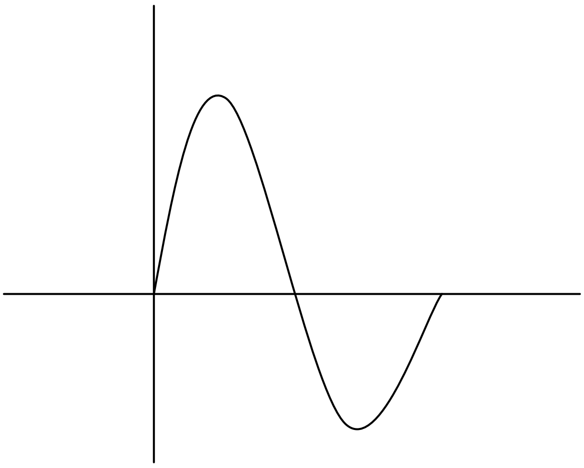
$$\frac{d^2y}{dx^2} = 6x$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$$

b. $y = \sin x$

D: $0 \le x \le 2\pi$

R: $-1 \le y \le 1$



Interceptes:

Set $y = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

$(0,0),(\pi,0),(2\pi,0)$ are x-int.

Set x=0 $\Rightarrow$ y=sin0=0

$\therefore$ (0,0) is y-int.

Symm.**:** w.r.t origin only

Asymptotes**:** No

$$\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = 0 \Rightarrow \cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ are critical points}$$

$$\frac{d^2y}{dx^2} = -\sin x$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow -\sin x = 0$$

$$x = 0, \pi, 2\pi$$

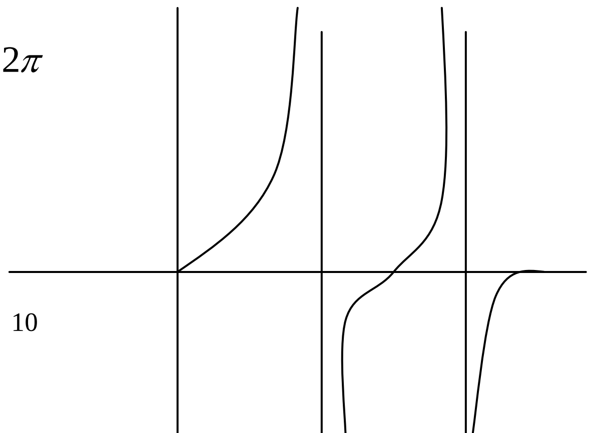
a. $y = \tan x$ for $0 \le x \le 2\pi$

$$y = \tan x = \frac{\sin x}{\cos x}$$

$$D: 0 \le x < \frac{\pi}{2} \; or \; \frac{\pi}{2} < x < \frac{3\pi}{2} \; or \; \frac{3\pi}{2} < x < 2\pi$$

$$R: -\infty < y < \infty$$

Intercepts:



Set y=0 $\Rightarrow$ tanx=0 $\Rightarrow$ x=0, π ,2 π

$\therefore$ (0,0),(π ,0),(2 π ,0) are x-int.

Set x=0 $\Rightarrow$ y=tan0=0

$\therefore$ (0,0) is y-int.

Symm **:** w.r.t origin

Asymptotes:

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ are V.Asy.

No H.Asy.

$\frac{dy}{dx} = \sec^2 x$

$\frac{dy}{dx} = 0 \Rightarrow \sec^2 x = 0$ impossible

$\frac{d^2y}{dx^2} = 2\sec^2 x \cdot \tan x$

$\frac{d^2y}{dx^2} = 0 \Rightarrow 2\sec^2 x \cdot \tan x = 0$

$\frac{\sin x}{\cos x} = 0$

$\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

b. $y = \ln x$

D: x>0

x=e^y R: all y

Intercepts:

Set y=0 $\Rightarrow$ ln x=0

x=e^0=1

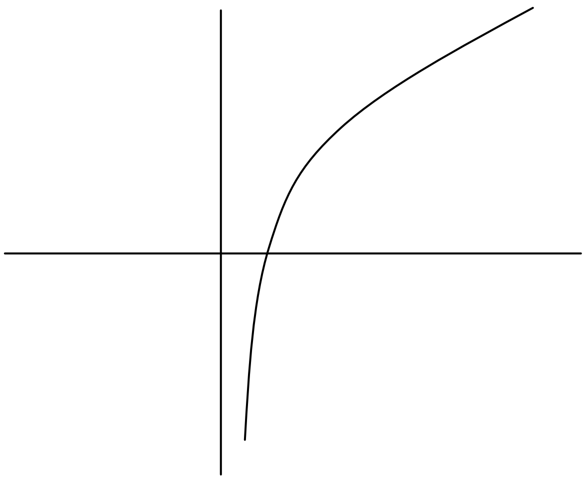
$\therefore$ (1,0)is x-int.

Set x=0 $\Rightarrow$ y=ln 0= ∞

No y-int.

Symm: No

Asymptotes: x=0 is V.Asy



$\frac{dy}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{x} = 0 \Rightarrow 1 = 0$

impossible

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$

$\frac{d^2y}{dx^2} = 0 \Rightarrow -\frac{1}{x^2} = 0 \Rightarrow -1 = 0$ impossible

c. $y=e^x$

D: all x

x=ln y   R: y>0

Interceptes:

Set x=0 $\Rightarrow$ $y=e^0=1$

$\therefore$ (0,1) is y-int.

Set y=0 $\Rightarrow$ $e^x=0$

x=ln 0 impossible

No x-int.

Symm : No

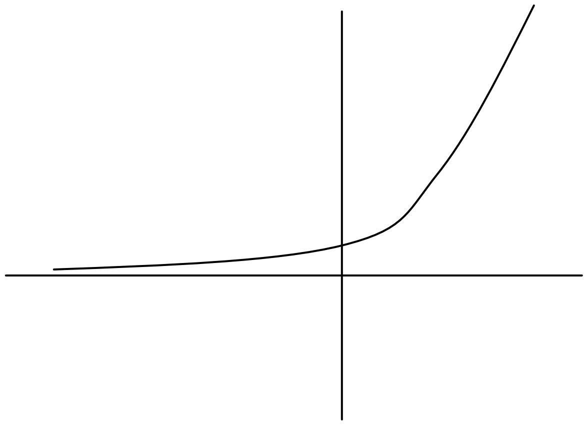
Asymptotes:

y=0 is H-Asy.

$y' = e^x \Rightarrow y' = 0 \Rightarrow e^x = 0$ impossible

$y'' = e^x$

$y'' = 0 \Rightarrow e^x = 0$ Impossible



Exercises:

Discuss and sketch the of the following functions

1. $y = x + \frac{1}{x}$
2. $y = \cos x$
3. $y = \ln(1-x)$
4. $y = e^{\frac{1}{2}x^2}$
5. $y = e^{-x}$
6. $y = 9x - x^2$
7. $y = 4x^3 - x^4$
8. $y = \frac{1}{x^2+3}$
9. $y = 2x(x+4)^3$

10. let $f(x) = |4 - x^2|$ on [-3,3] find the local and absolute maxima and minima.

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